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GAUSSIAN RANDOM PROCESSES. PART 2.(U)

NOV 77 I A IBRAGIMOV, Y A ROZANOV

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FOREIGN TECHNOLOGY DIVISION

GAUSSIAN RANDOM PROCESSES

by

I. A. Ibragimov, Yu. A. Rozanov

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Chapter of V.

~~FULL/TOTAL/~~COMPLETE REGULARITY. Processes with discrete time.

§1. Determinations. Preliminary constructions. Examples.

In this chapter we will examine stationary in the broad sense process ξ (t) with the discrete time t = 0, $\pm$ 1,.... Everywhere here we deal only with such concepts, which are formulated in terms of the first two torque/moments, so that it is indifferent, is process ξ (t) Gaussian or not.

Recall (see Chapter IV) that the process ξ (t) is called completely regular, if the coefficient of the regularity

$$\rho(\tau)=\sup_{\eta_1\in H(\tau,\infty),\ \eta_2\in H(-\infty,0)}|M\eta_1\bar{\eta}_2|=\sup|(\eta_1,\eta_2)|\underset{\tau\to\infty}{\to}0$$

(sup is taken for η_1, η_2, by that satisfying the condition of standardization $\|\eta_1\|=\|\eta_2\|=1$).

In the present chapter we will trace the properties of spectral characteristics of completely regular processes ξ (t). For this it is first of all necessary to rewrite expression for the coefficient of regularity in spectral form.

Let us recall that any completely regular process (linearly) is regular, and therefore

a) it has spectral density f (λ), representable in the form

$$f(\lambda)=|g(e^{i\lambda})|^2, \qquad (1.1)$$

where g is a function of Hardy's class $\mathscr{H}^2$ in unit circle;

$$b) \int_{-\pi}^{\pi}|\ln f(\lambda)\,d\lambda|<\infty. \qquad (1.2)$$

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(In reality (1.1) and (1.2) are equivalent and

$$g(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{i\lambda}+z}{e^{i\lambda}-z}\ln f(\lambda)\,d\lambda\right\},\ |z|<1.)$$

Hence and from theorem 1 chapter II immediately follow that

$$\rho(\tau)=\sup_{\varphi,\psi}\left|\int_{-\pi}^{\pi}e^{i\lambda\tau}\varphi(e^{i\lambda})\psi(e^{i\lambda})f(\lambda)\,d\lambda\right|=\sup_{\varphi,\psi}|\langle e^{i\lambda\tau}\varphi,\ \bar{\psi}\rangle_F| \qquad (1.3)$$

where sup it is taken according to all functions $\varphi(e^{i\lambda})$, $\psi(e^{i\lambda})$, that belongs to subspace $L^+(F)=\frac{1}{g}\mathscr{H}^2$, which they satisfy the condition of the standardization

$$\|\varphi\|_F = \left(\int_{-\pi}^{\pi} |\varphi(e^{i\lambda})|^2 f(\lambda)\, d\lambda \right)^{1/2} = 1, \; \|\psi\|_F =$$

$$= \left(\int_{-\pi}^{\pi} |\psi(e^{i\lambda})|^2 f(\lambda)\, d\lambda \right)^{1/2} = 1. \qquad (1.4)$$

It is useful to note that value ρ (τ) will not be changed, if we in (1.3) take sup on any dense in $L^+(F)$ multitude of functions, which satisfy (1.4), for example according to polynomials or functions from $\mathscr{H}^2$.

If $\varphi, \psi \in \mathscr{H}^2$, that $\theta = \varphi\psi \in \mathscr{H}^1$, a $\|\theta\|_F^{(1)} \leqslant \|\varphi\|_F \cdot \|\psi\|_F$. On the contrary, if function $\theta \in \mathscr{H}^1$, it it is possible to register in the form (see §1 chapter II) of product $\theta = \varphi\psi$ two functions $\varphi, \psi \in \mathscr{H}^2$, whereupon for almost all λ $|\varphi| = |\psi| = |\theta|^{1/2}$, and, consequently, $\|\theta\|_F^{(1)} = \|\varphi\|_F = \|\psi\|_F$.

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Therefore besides (1.3) occurs the still following expression for ρ (τ):

$$\rho(\tau) = \sup_{\theta} \left| \int_{-\pi}^{\pi} e^{i\lambda\tau} \theta(\lambda) f(\lambda)\, d\lambda \right|, \qquad (1.5)$$

where sup it is taken on all $\theta \in \mathcal{H}^1, \|\theta\|_F^{(1)} \leqslant 1$.

In order to obtain one additional necessary subsequently expression for $\rho(\tau)$, let us be based on the theorem of the B'yerling, according to which the set of functions $\{\varphi g\}$, where $\phi(z)$ passes all polynomials, is dense in space $\mathcal{H}^2$. By beginning from (1.3), we let us find that

$$\rho(\tau) = \sup_{\varphi, \psi} \left| \int_{-\pi}^{\pi} \varphi(e^{i\lambda}) \psi(e^{i\lambda}) e^{i\lambda\tau} f(\lambda)\, d\lambda \right| = \sup_{\varphi, \psi} \left| \int_{-\pi}^{\pi} (\varphi g)(\psi g) e^{i\lambda\tau} \frac{\bar{g}}{g}\, d\lambda \right| = \sup_{\varphi_1, \psi_1} \left| \int_{-\pi}^{\pi} \varphi_1(e^{i\lambda}) \psi_1(e^{i\lambda}) e^{i\lambda\tau} \frac{\overline{g(e^{i\lambda})}}{g(e^{i\lambda})}\, d\lambda \right|, \quad (1.6)$$

whereupon in the last/latter integral sup it is taken on all from the single sphere of space $\mathcal{H}^2$.

Finally, in the same way as of (1.3) we were obtained (1.5), from (1.6) it is possible to deduce the equality

$$\rho(\tau) = \sup_{\theta} \left| \int_{-\pi}^{\pi} \theta(e^{i\lambda}) e^{i\tau\lambda} \frac{\overline{g(e^{i\lambda})}}{g(e^{i\lambda})} d\lambda \right|, \qquad (1.7)$$

where this time θ it passes the single sphere of space $\mathcal{H}^1$.

Below everywhere in this chapter will be encountered only the absolutely continuous spectral functions, which are assigned by their (spectral) densities. In order not to introduce excess designations, subsequently for the extent/elongation of an entire chapter let us use symbols $L(f)$, $L^+(g)$, $\|\cdot\|_h$, $\langle\cdot,\cdot\rangle_f$ and so forth instead of $L(F)$, $L^+(G)$, $\|\cdot\|_H$, $\langle\cdot,\cdot\rangle_F$ and so forth, where f = F', g = G', h = H' and so forth.

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Being returned to our problem of the description of completely regular processes, let us note that it it is possible to reformulate as follows: to describe the class of the nonnegative summarized on [$-\pi$, π] functions f (λ), for which the determined by equalities (1.3) or (1.5) values ρ (τ) approach 0 with $\tau \to \infty$. This analytical formulation is the basis of our further research.

Now before passing to more detailed analysis, let us give several assertions, which almost immediately escape/ensue from (1.3) or (1.5), but which nevertheless will make it possible to obtain the

general idea of the structure of the spectral densities of completely regular processes.

First of all, by set/assuming in (1.5) θ (λ) $\equiv$ 1, we will find that the n Fourier coefficient f (λ) satisfies the inequality

$$|B(n)|\leqslant\rho(n)\int_{-\pi}^{\pi}f(\lambda)\,d\lambda.$$

Consequently, the condition of full/total/complete regularity assigns some limitations on smoothness f (λ). So, if $\sum\rho(n)<\infty$, then f (λ) is continuous. It is below, improving this very rough approach, we will obtain considerably more powerful results.

From inequality (1.2) it follows that f (λ) cannot have zero extremely high orders. Below (§5) it will be shown, that in reality on zero f (λ) are assigned much more severe limitations (speaking in general terms, f (λ) is a product of function without zeros by the square modulus of polynomial).

On the contrary, if spectral density f (λ) sufficiently smooth and positive, the corresponding random process completely is regular. Specifically, occurs the following.

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Theorem 1. If spectral density f (λ) is continuous [1] and strictly positive, f (λ) $\geqslant$ m > 0, then corresponding f (λ) stationary process is completely regular, whereupon

$$\rho(\tau) \leqslant \frac{1}{m} E_{\tau-1}(f). \quad (1.8)$$

FOOTNOTE [1]. Recall that the points π and π are identified. ENDFOOTNOTE.

Here and throughout through $E_n(h)$ is designated the value of the best approximation of function h (λ) by the trigonometric polynomials of degree not higher than n on segment [-π, π] in uniform metric.

Actually, for any trigonometric polynomial θ (λ) degrees are not above τ-1 and any function $0 \in \mathscr{H}^1$

$$\int_{-\pi}^{\pi} e^{i\lambda\tau}\theta(\lambda)\, Q(\lambda)\, d\lambda = 0.$$

Therefore, if P (λ) - the polynomial of the best approximation for f of degree $\leqslant$τ-1, then on the strength of (1.5)

$$\rho(\tau) = \sup_{\theta}\left|\int_{-\pi}^{\pi} e^{i\lambda\tau}\theta(\lambda)\,[f(\lambda) - P(\lambda)]\, d\lambda\right| \leqslant$$

$$\leqslant E_{\tau-1}(f) \sup_{\|\theta\|_f^{(1)}=1}\left|\int_{-\pi}^{\pi} |\theta(\lambda)|\, d\lambda\right| \leqslant \frac{1}{m} E_{\tau-1}(f).$$

Limitation f $\geqslant$ m can be attenuate/weakened, after noting that occurs the following common/general/total result.

Theorem 2. If w (λ) there is the spectral density of completely regular process, and P (z) - the polynomial of degree n, then u

$$f(\lambda) = |P(e^{i\lambda})|^2 w(\lambda) \qquad (1.9)$$

there is the spectral density of completely regular process. In this case

$$\rho(\tau; f) \leqslant \rho(\tau - n; w). \qquad (1.10)$$

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Actually, $e^{i\lambda n}|P|^2 \in \mathscr{H}^1$. Furthermore, $\|\theta P\|^{(1)}_{w} = \|\theta\|^{1}_{f}$. Therefore

$$\rho(\tau; f) = \sup_{\theta} \left| \int_{-\pi}^{\pi} e^{i(\tau-n)\lambda} \theta(\lambda) [e^{in\lambda} |P(\lambda)|^2] w(\lambda) d\lambda \right| \leqslant$$

$$\leqslant \sup_{\|\theta\|^{(1)}_{w} \leqslant 1} \left| \int_{-\pi}^{\pi} e^{i(\tau-n)\lambda} \theta(\lambda) w(\lambda) d\lambda \right| = \rho(\tau - n; w).$$

Theorem is demonstrated. Below we frequently will encounter the expansions of form (1.9).

§2. First method of study. The theorem of Khelsona is and Sarasana.

Here we is presented the belonging Khelsonu and Sarasanu [1] the description of the set of the spectral densities of completely regular processes.

FOOTNOTE [1]. H Helson, Sarasan, Past and Future, Math. Scand. 21, No. 1 (1967), 5-16. ENDFOOTNOTE.

Their method is substantially connected with output/yield into

composite plane. In connection with this to us it is convenient to consider that spectral density f (λ) is assigned on circumference $C: |z| = 1,\ z = re^{i\lambda}$.

Theorem 3. Stationary process ξ (t) with discrete time is completely regular then and only then, if it has spectral density f (λ), representable in the form

$$f(\lambda) = w(\lambda) \mid P(e^{i\lambda}) \mid^2. \tag{2.1}$$

Here P (z) - polynomial with roots on $|z| = 1$, and function w (λ) with any $\varepsilon > 0$ representable as

$$w = \exp\{r_\varepsilon + u_\varepsilon + \tilde{v}_\varepsilon\}, \tag{2.2}$$

where r_ε is continuous on C, $a \| u_\varepsilon \|^{(\infty)} + \| v_\varepsilon \|^{(\infty)} \leqslant \varepsilon$.

Proof. Sufficiency. From theorem 2 it follows that it is possible to count P $\equiv$ 1.

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Relying on the Weyerstrass theorem, let us select trigonometric polynomial Q_ε degree r_0 so, in order to

$$e^{r_\varepsilon} = Q_\varepsilon (1 + \theta), \ \max_\lambda | \theta(\lambda) | < \varepsilon.$$

Then (P $\equiv$ 1!)

$$f(\lambda) = e^{r_\varepsilon + u_\varepsilon + \tilde{v}_\varepsilon} = Q_\varepsilon e^{\tilde{v}_\varepsilon - i v_\varepsilon} (1 + \theta_\varepsilon), \tag{2.3}$$

where $\| \theta_\varepsilon \|^{(\infty)} \leqslant 7\varepsilon$, if $\varepsilon < 1$.

Let us assume $f_\varepsilon = |Q_\varepsilon||e^{\vartheta_\varepsilon - iv_\varepsilon}|$. On the strength of (2.3)

$$\frac{1}{2}\|\varphi\|_{f_\varepsilon} \leqslant \|\varphi\|_f \leqslant 2\|\varphi\|_{f_\varepsilon},$$

if only ε is sufficiently small. Further, function $e^{i\lambda\tau_0}Q_\varepsilon(\lambda)e^{\vartheta_\varepsilon(\lambda) - iv_\varepsilon(\lambda)}$ is summarized and is the boundary value (on circumference $|z| = 1$) of the function, analytical in circle $|z| < 1$. Consequently, for any two polynomials $\varphi(z)$, $\psi(z)$ and all $\tau > \tau_0$

$$\int_{-\pi}^{\pi} \varphi(e^{i\lambda})\psi(e^{i\lambda})e^{i\lambda\tau}Q_\varepsilon(\lambda)e^{\vartheta_\varepsilon - iv_\varepsilon}\,d\lambda = 0.$$

Therefore for all $\tau > \tau_0$ on the strength of (2.3)

$$\rho(\tau) = \sup_{\varphi,\psi}\left|\int_{-\pi}^{\pi}\varphi(e^{i\lambda})\psi(e^{i\lambda})e^{i\lambda\tau}f(\lambda)\,d\lambda\right| \leqslant$$
$$\leqslant \sup_{\varphi,\psi}\int_{-\pi}^{\pi}|\varphi(e^{i\lambda})||\psi(e^{i\lambda})||f_\varepsilon(\lambda)||\theta_\varepsilon(\lambda)|\,d\lambda \leqslant$$
$$\leqslant 7\varepsilon\|\varphi\|_{f_\varepsilon}\|\psi\|_{f_\varepsilon} \leqslant 28\varepsilon. \qquad (2.4)$$

The sufficiency of the conditions of theorem is demonstrated.

Need. First, being transmitted of equality (1.7), let us

demonstrate the following lemma.

Lemma 1. The coefficient of regularity satisfies the equality

$$\rho(\tau)=\inf_A\left\|\frac{\bar{g}}{g}e^{i\lambda\tau}-A\right\|^{(\infty)}, \qquad (2.5)$$

where inf it is taken according to all functions $A\in\mathscr{H}^{\infty}$, $\dot{A}(0)=0$.

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The assertion of lemma is a special case of the common/general/total duality principle for analogous extreme problems, at basis of which lie/rests the well known theorem of the Khan - Banach.

Actually, the integral in right side (1.7) determines the linear functional l (θ) in $\mathscr{L}^1$. Examine/considered only in $\mathscr{H}^1\subset\mathscr{L}^1(-\pi,\pi)$, this functional (in accordance with (1.7)) has a norm, equal to ρ (τ). Any continuation l_1 (θ) this functional l (θ) from space $\mathscr{H}^1$ to everything $\mathscr{L}^1$ takes the form

$$l_1(\theta)=l(\theta)-l^*(\theta),$$

where l^* (θ) it is converted into zero on $\mathscr{H}^1$. All the continuations have a norm, not less ρ (τ); according to the theorem of the Khan - Banach among them there exists continuation l_1 with norm, in accuracy equal to ρ (τ). Consequently,

$$\rho(\tau)=\inf_{l^*}\|l-l^*\|,$$

whereupon in this equality symbol $\|\cdot\|$ designates the norm of functional in $\mathscr{L}^1$. Any functional L in $\mathscr{L}^1(-\pi,\pi)$ according to known theorem is represented in the form

$$L(\theta)=\int_{-\pi}^{\pi}\theta(e^{i\lambda})\,A(e^{i\lambda})\,d\lambda,\qquad A\in\mathscr{L}^{\infty},\quad \|L\|=\|A\|^{(\infty)}.$$

Requirement l * (θ) = 0 for all $\theta\in\mathscr{H}^1$ is equivalent to the fact that for function $A\in\mathscr{L}^{\infty}(-\pi,\pi)$, determining functional l *

$$\int_{-\pi}^{\pi}A(e^{i\lambda})\,e^{in\lambda}\,d\lambda=l^{*}(e^{in\lambda})=0,\qquad n\geqslant 0.$$

These equalities in turn, are equivalent to the fact that $A\in\mathscr{H}^{\infty}$ and A (0) = 0. Hence and from (1.7) it follows that

$$\rho(\tau)=\inf_{l^{*}}\|l-l^{*}\|=\inf_{A\in H^{\infty},\,A(0)=0}\left\|\frac{\bar{g}}{g}e^{i\tau\lambda}-A\right\|^{\infty}.$$

Lemma is proved. From it almost immediately it follows.

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Lemma 2. For the full/total/complete regularity of process $\xi(t)$ it is necessary and sufficiently in order that on any $\varepsilon>0$ would be located function $A\in\mathscr{H}^{\infty}$ such which is simultaneous

$$-\varepsilon<\ln|A|<\varepsilon,\qquad -\varepsilon<\arg(Ag^{2}e^{-i\tau\lambda})<\varepsilon\ \ (\mathrm{mod}\ 2\pi).\ (2.6)$$

Actually, $\bar{g}/g = \exp\{-i\arg(g^2)\}$. Therefore (2.5) it is possible to register in the form

$$\rho(\tau) = \inf_A \| 1 - e^{\ln|A|} \exp\{-i\arg(Ag^2e^{-i\tau\lambda})\} \|^\infty,$$

whence in view $\rho(\tau) \rightarrow 0$ and it follows (2.6).

Us will be required one additional lemma about analytical continuation; the use of this lemma and duality principle composes the nucleus of the proof of Khelsona and Sarasana.

Lemma 3. Let function S (z) be analytical in circle $|z| < 1$, excluding point z = 0, where it can have a pole of order r. If $z^r S(z) \in \mathscr{H}^{1/2}$. and on $|z| = 1$ function S is real and nonnegative, it analytically continuable through $|z| = 1$ in $|z| > 1$, and continued function is a polynomial of z, 1/z.

We will plot the proof of lemma to the end of the paragraph, and thus far, by assuming lemma demonstrated, let us finish the proof of theorem 3.

Relying on (2.6), let us select function $s(e^{i\lambda}) \in \mathscr{L}^\infty$ so, in order to

$$|s| \leqslant \varepsilon, \ \arg(Ag^2e^{-i\tau\lambda}) + s \equiv 0 \pmod{2\pi}.$$

Let us consider the function

$$S(z) = A(z)\, g^2(z)\, z^{-\tau} e^{(is - \tilde{s})(z)}.$$

It is obvious, function $z^{\tau} S(z)$ is analytical in $|z| < 1$; on $|z| = 1$

$$S(e^{i\lambda}) = |A(\lambda)|\, e^{-\tilde{s}(e^{i\lambda})} f(\lambda) \geqslant 0. \qquad (2.7)$$

It is well known that the limitations on an increase in function s (λ) assign the appropriate limitations on increase adjoint function $\tilde{s}$.

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Specifically, from inequality $|s| \leqslant \varepsilon$ it follows (see [13], page 404), that $\exp(k|\tilde{s}|) \in \mathscr{L}^1(-\pi, \pi)$ for all $k < \frac{\pi}{2\varepsilon}$; consequently, for $\varepsilon < 2/\pi$

$$\int_{-\pi}^{\pi} |S(e^{i\lambda})|^{1/2} d\lambda < \sqrt{2} \left(\int_{-\pi}^{\pi} f(\lambda)\, d\lambda \int_{-\pi}^{\pi} e^{|\tilde{s}(e^{i\lambda})|}\, d\lambda \right)^{1/2} < \infty.$$

Thus, for function S (z) are satisfied all conditions of lemma 3; therefore this function is polynomial of z and 1/z.

Further, on circumference $|z| = 1$ function S (z) is a nonnegative trigonometric polynomial. According to Feyera - Riesz' theorem nonnegative trigonometric polynomial is a square modulus of polynomial of $e^{i\lambda}$ (see [11], page 33). So that

$$S(e^{i\lambda}) = |P_1(e^{i\lambda})|^2,$$

where P_1 (z) - polynomial. Let us register P_1 in the form $P_1 = P \cdot Q$ where Q and P - polynomials with the roots, arrange/located respectively outside by $|z| = 1$ and by $|z| = 1$. From equality (2.7) it follows then that

$$\begin{aligned} f(\lambda) &= S(e^{i\lambda}) |A|^{-1} e^{\tilde{s}} = \\ &= |P(e^{i\lambda})|^2 \exp\{\ln|Q|^2 - \ln|A| + \tilde{s}\} = \\ &= |P|^2 \exp\{r_\varepsilon + u_\varepsilon + \tilde{v}_\varepsilon\}, \end{aligned} \tag{2.8}$$

where 1) $r_\varepsilon = \ln|Q|^2$ - is a continuous function; 2) $u_\varepsilon = -\ln|A|$ and $\|u_\varepsilon\|^{(\infty)} \leqslant \varepsilon$; 3) $v_\varepsilon = s$ also, regarding s $\|v_\varepsilon\|^{(\infty)} \leqslant \varepsilon$.

In order to finish the proof of theorem to us remained to check that polynomial P does not depend on ε. Let

$$f(\lambda) = |P'|^2 \exp\{r_{\varepsilon'} + u_{\varepsilon'} + v_{\varepsilon'}\}. \tag{2.9}$$

Let us show that polynomials P and P' coincide with an accuracy to constant factor. Above we already noted that $e^{|\tilde{v}_\varepsilon|} \in \mathscr{L}^1$, $e^{|\tilde{v}_{\varepsilon'}|} \in \mathscr{L}^1$.

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Therefore all the functions $\left(\frac{f}{|P|^2}\right)^{\pm 1}$, $\left(\frac{f}{|P'|^2}\right)^{\pm 1}$ are summarized. On Schwarz inequality

$$\int_{-\pi}^{\pi}\left|\frac{P}{P'}\right|d\lambda \leqslant \left(\int_{-\pi}^{\pi}\frac{|P|^2}{f}\,d\lambda\int_{-\pi}^{\pi}\frac{f}{|P'|^2}\,d\lambda\right)^{1/2} < \infty,$$
$$\int_{-\pi}^{\pi}\left|\frac{P'}{P}\right|d\lambda \leqslant \left(\int_{-\pi}^{\pi}\frac{|P'|^2}{f}\,d\lambda\int_{-\pi}^{\pi}\frac{f}{|P|^2}\,d\lambda\right)^{1/2} < \infty. \tag{2.10}$$

Inequalities (2.10) mean that polynomials P and P' divide each other, roots of both of polynomials lie/rest into $|z| = 1$ and, therefore, P/P' = const.

Theorem 3 is demonstrated; there remained only to be convinced of the validity of lemma 3. Function $z^{\tau}S \in \mathscr{H}^{1/2}$ can be registered in the form of product $z^{\tau}S = b\theta_1\theta_2$, where b - produced to Bliyashke, θ_1 - the internal function, analytical in $|z| < 1$, θ_2 - external function in $|z| < 1$. On $|z| = 1$ $|b| = |\theta_1| = 1$; function $\theta_2^{1/2} \in \mathscr{H}^1$. Therefore, by set/assuming $S_1 = z^{-\tau}b\theta_1\theta_2^{1/2}$, $S_2 = \theta_2^{1/2}$, let us present S in the form of product S_1S_2, where $z^{\tau}S_1$, $S_2 \in \mathscr{H}^1$ and with $|z| = 1$ $|S_1| = |S_2|$. Hence and from matter S on circumference $|z| = 1$ it follows that (on $|z| = 1$) $S_1 = \bar{S}_2$. Consequently, on

circumference $|z| = 1$ function $S_1 + S_2$ and i $(S_1 - S_2)$ are real; they are summarized on $|z| = 1$ and by the known principle of the symmetry analytically are extendible through $|z| = 1$ (values of the continued functions for $|z| > 1$ are determined by equalities $(S_1 + S_2)(z) = \overline{(S_1 + S_2)(\bar{z}^{-1})}$, $i(S_1 - S_2)(z) = \overline{i(S_1 - S_2)(\bar{z}^{-1})}$ [1].

FOOTNOTE [1]. For the proof of the analyticity of the continued functions in the vicinity of circumference $|z| = 1$ one should use the valid for all functions $\gamma \in \mathscr{H}^1$ equality

$$\lim_{r \uparrow 1} \int_{\alpha}^{\beta} \gamma(re^{i\lambda})\, d\lambda = \int_{\alpha}^{\beta} \gamma(e^{i\lambda})\, d\lambda.$$

ENDFOOTNOTE.

But then also function S_1, S_2 (but that means and $S = S_1 S_2$) analytically are extendible through $|z| = 1$.

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In this case of the very method of continuation it follows that the continued function S is analytical everywhere, with the exception perhaps only of points $z = 0$, $z = \infty$, where it can have poles of order not above v. According to the theorem of Liuvillya S (z) it is

necessary to eat a polynomial of z and 1/z. Lemma 3 is demonstrated. Thereby is finished the proof of theorem 3.

From the demonstrated theorem escape/ensues a series of the simple corollaries, which make it possible to obtain the sufficiently demonstrative representation of the structure of the spectral densities of completely regular processes. The conclusion/derivation of these corollaries we will plot to §5.

The examined here method makes it possible also completely to describe the class of those random processes, for which ρ (1) < 1 (but not compulsorily $\rho (\tau) \rightarrow 0$).

As already mentioned, inequality ρ (1) < 1 also is a condition of regularity, more powerful than linear regularity, and indicating geometrically, that the minimum angle between subspaces $L^{+}, e^{i\lambda}L^{-}$, was positive. We will obtain below deeper result [1], after indicating the conditions, by which $\rho (\tau) < 1$ for the first time with $\tau = k$.

FOOTNOTE [1]. This result with k = 1 is establish/installed Khelsonom and to Sege (H Helson, G Szego, A problem in prediction tneory, Ann. Math. Pure Appl. 51 (1960), 107-138), with k > 1 - Khelsonom and Sarasanom (see reference on page 209). ENDFOOTNOTE.

Theorem 4. For stationary process $\xi(t)$, $t = 0, \pm 1, \ldots$, the relationship/ratios

$$\rho(k-1)=1, \quad \rho(k)<1$$

are fulfilled in that and only that case, if it has spectral density $f(\lambda)$, representable in the form

$$f(\lambda)=|P(e^{i\lambda})|^2 e^{u+\tilde{v}(\lambda)},$$

where P (z) - the polynomial of degree k-1 with roots on $|z| = 1$, and u (λ), v (λ) - the real bounded functions, whereupon $\|v\|^{(\infty)}<\pi/2$.

Let us demonstrate first the sufficiency of the conditions indicated. On the strength of (1.10)

$$\rho(k;\ f)\leqslant\rho(1;\ e^{u+\tilde{v}}),$$

it is possible to be bounded to the examination of that case, when $P \equiv 1$. It is obvious, $\ln f=u+\tilde{v}\in\mathscr{L}^1$, so that $f(\lambda)=|g(e^{i\lambda})|^2$, where $g\in\mathscr{H}^2$.

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Let A (z) - the external function of class $\mathscr{H}^\infty$, for which $|A(e^{i\lambda})|=e^{-u}$. Let us construct, further, function $\psi=e^{\tilde{v}-iv}$. This is the external function of class $\mathscr{H}^1$, since, as noted above, $e^{\tilde{v}}\in\mathscr{L}^1$, as soon as $|\tilde{v}|<\pi/2$. On $|z|= 1$

occurs the equality $|\psi|=|Ag^2|$. The external functions whose module/moduli to $|z|=1$ coincide, differ perhaps only in terms of constant factor, so that it is possible to count $\psi=Ag^2$.

Let us select positive number γ so, in order to $\gamma|A|\leqslant 1$. The values of function $\zeta=\gamma e^{-u}e^{-iv}$ with all λ lie/rest at region $\mathfrak{A}=\{0<\gamma\inf\limits_{\lambda}e^{-u}\leqslant|\zeta|\leqslant 1,\ |\arg\zeta|<\sup\limits_{\lambda}|v|<\pi/2\}$. It is not difficult to count, that $\rho_1=\inf\limits_{\zeta\in\mathfrak{A}}|1-\zeta|<1$. But then on the strength of (2.5) and

$$\rho(1)\leqslant\left\|\frac{g}{\bar g}-\gamma A\right\|^{(\infty)}=\|1-\gamma e^{\ln|A|}\exp\{-i\arg(Ag^2)\}\|^{(\infty)}=$$
$$=\|1-\gamma e^{-u}e^{-iv}\|^{(\infty)}\leqslant\rho_1<1.$$

The sufficiency of the conditions of theorem is establish/installed. Let us demonstrate need.

Recall that the inequality $\rho(k)<1$ will draw the regularity of process $\xi(t)$. Therefore spectral density f is factorable: $f=|g|^2$, $g\in\mathcal{H}^2$, and $\rho(k)$ is computed with the help of formula (2.5). On the basis of this formula, it is analogous with lemma 2, is concluded, that as soon as $\rho(k)<1$, necessary will be located function $A\in\mathcal{H}^\infty$ with the properties

$$|A|\geqslant\varepsilon,\quad|\arg(Ag^2e^{-i(k-1)\lambda})|\leqslant\frac{\pi}{2}-\varepsilon,\quad\varepsilon>0$$

(second inequality occurs on mod 2π).

Let us determine function v, $|v|\leqslant\frac{\pi}{2}-\varepsilon$, so, in order to

$$v+\arg(Ag^2e^{-i(k-1)\lambda})\equiv 0\pmod{2\pi}.$$

The function

$$S(z) = Ag^2 z^{-(k-1)} e^{-\tilde{v}+iv}$$

is analytical in $|z| < 1$, excluding the pole of order k - 1 in zero. It is analogous with that, as this is done on page 213, is concluded, that $Sz^{k-1} \in \mathscr{H}^{1/2}$, and since $S = |P|^2$, where P polynomial of degree are not above k-1.

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Finally, set/assuming $u = -\ln|A|$ let us find that

$$f = |P|^2 e^{u+\tilde{v}}.$$

It remained to show that degree N of polynomial P, not exceeding k-1, in reality was equal to k-1. On already demonstrated $\rho(1; e^{u+\tilde{v}}) < 1$. Therefore, if N < k-1, then

$$\rho(k-1;\ f) \leqslant \rho(k-1-N;\ e^{u+\tilde{v}}) \leqslant \rho(1;\ e^{u+\tilde{v}}) < 1,$$

in spite of equality ρ (k-1; f) = 1. Theorem is demonstrated.

§3. Second method of research. Local conditions (it began).

In this and following paragraph we is presented new approach to

research of completely regular sequences. Unlike methods §2, it is purely real and leads to local conditions for f (λ). However, there is a discontinuity/interruption between the necessary and sufficient conditions in the given here local form.

Theorem 5. In order that positive summarize on [-π, π] function f (λ) would be the spectral density of completely regular stationary sequence, it is necessary that it would be represented in the form

$$f(\lambda) = |P(e^{i\lambda})|^2 w(\lambda), \qquad (3.1)$$

where P (z) - polynomial with roots on $|z| = 1$, and original W (λ) of function w (λ) satisfies the condition

$$\lim_{\delta \to 0} \omega_W(\delta) = 0, \qquad (3.2)$$

where

$$\omega_W(\delta) = \sup_{\lambda} \sup_{|x| \leqslant \delta} \frac{|W(\lambda + x) + W(\lambda - x) - 2W(\lambda)|}{|W(\lambda + x) - W(\lambda - x)|}.$$

One of the simplest inversions of this theorem has the following form.

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Theorem 6. Let spectral density f (λ) of sequence ξ (t) is representable in the form (3.1), where P (z) - the polynomial of degree k, and of w (λ) possesses the following properties:

1) $\sum_n \omega_W^2(2^{-n}) < \infty$;

2) $0 < m \leqslant w(\lambda) \leqslant M < \infty$.

Then sequence $\xi(t)$ is completely regular, whereupon

$$\rho(\tau) \leqslant 40\left(\frac{M}{m}\right)^{1/2}\left(\sum_1^\infty \omega_W^2\left(\frac{1}{2^{n-1}(\tau-k)-1}\right)\right)^{1/2}. \quad (3.3)$$

The proof of theorem 5 is sufficiently cumbersome. It based on the analysis of the functions

$$\gamma(N;\mu) = \int_{-\pi}^{\pi} \frac{\sin^2 N\frac{\lambda-\mu}{2}}{\sin^2\frac{\lambda-\mu}{2}} f(\lambda)\,d\lambda,$$

which is carried out further in this paragraph [1].

FOOTNOTE [1]. Our research of function γ (N; μ) in many respects they follow V. P. Leonov's work "on the dispersion of the time/temporary average of stationary random process", is theoretically probable. and it applicable VI, No 1 (1961), 93-101. ENDFOOTNOTE.

In the following paragraph, being based on this analysis, we will demonstrate theorem 5. There will be demonstrated theorem 6.

Passing to research γ (N: μ), let us note first that

$$\gamma(N;\mu)=\mathrm{M}\left|\sum_{0}^{N-1}e^{i\mu t}\xi(t)\right|^{2}. \qquad (3.4)$$

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Actually, if the spectral representation of sequence ξ (t) takes the form

$$\xi(t)=\int_{-\pi}^{\pi}e^{i\lambda t}\Phi(d\lambda)$$

(where $\Phi(d\lambda)$ – is an orthogonal random measure, $\mathrm{M}|\Phi(d\lambda)|^{2}=f(\lambda)\,d\lambda$), then

$$\mathrm{M}\left|\sum_{0}^{N-1}e^{i\mu t}\xi(t)\right|^{2}=\mathrm{M}\left|\int_{-\pi}^{\pi}\frac{e^{iN(\lambda-\mu)}-1}{e^{i(\lambda-\mu)}-1}\Phi(d\lambda)\right|^{2}=$$

$$=\int_{-\pi}^{\pi}\frac{\sin^{2}N\frac{\lambda-\mu}{2}}{\sin^{2}\frac{\lambda-\mu}{2}}f(\lambda)\,d\lambda.$$

Lemma 4. At N $\rightarrow \infty$ value

$$\gamma(N; 0) = \gamma(N) = \mathsf{M}\left|\sum_{1}^{N} \xi(t)\right|^2 = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda)\, d\lambda$$

either approaches ∞ or is limited. The latter occurs in that and only that case, if

$$\int_{-\pi}^{\pi} \frac{f(\lambda)}{\sin^2 \frac{\lambda}{2}}\, d\lambda < \infty. \qquad (3.5)$$

Proof. Let U designate unitary operator in space H = H $(-\infty, \infty)$, that corresponds to sequence ξ (t): U (t) = ξ (t + 1). Let us consider sums $S_N = \sum_0^{N-1} \xi(t)$ as cell/elements of space H. Assertion $\varliminf_N \mathsf{M} | S_N |^2 < \infty$ means that for certain subsequence $N_j \| S_{N_j} \| = \mathsf{M}^{1/2} | S_{N_j} |^2 <$ C < ∞. Since sphere in hilbert space H weakly is compact, from $\{N_j\}$ it is possible to extract this subsequence $\{n_k\}$, which S_{n_k} weakly converge to certain cell/element $-\eta \in H$ i.e., for all $\zeta \in H$

$$\lim_{n_k} (S_{n_k}, \zeta) = \lim_{n_k} \mathsf{M} S_{n_k} \zeta = -(\eta, \zeta) = -\mathsf{M}\eta\zeta.$$

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But then

$$\lim_{n_k} (US_{n_k}, \zeta) = -(U\eta, \zeta),$$

and, therefore, with all $\xi \in H$

$$(U\eta - \eta, \zeta) = \lim_{n_k} (S_{n_k} - US_{n_k}, \zeta) = (\xi(0), \zeta) - \lim_{n_k} (\xi(n_k), \zeta) = M\xi(0)\bar{\zeta} - \lim_{n_k} M\xi(n_k)\bar{\zeta}.$$

On the strength of the isometry between H and L (f) random variable ζ answers the function $\varphi \in L(f)$ such, that

$$M\xi(n_k)\bar{\xi} = \int_{-\pi}^{\pi} e^{i\lambda n_k} \overline{\varphi(\lambda)} f(\lambda)\, d\lambda. \tag{3.6}$$

It is obvious, $\bar{\varphi} f \in \mathscr{L}^1(-\pi, \pi)$, and in terms of rimana - Lebesgue's theorem integral in right side (3.6) vanishes, when $n_k \to \infty$. Therefore, if lim γ (N) < ∞, then with all $\zeta \in H$

$$M(U\eta - \eta)\bar{\zeta} = M\xi(0)\bar{\zeta},$$

and, therefore,

$$\xi(0) = U\eta - \eta = \eta(1) - \eta(0),$$

where $\eta(0) = \eta$, $\eta(k) = U^k\eta$. But in that case $\xi(t) = \eta(t+1) - \eta(t)$ and with all N

$$S_N = \eta(N) - \eta(0),$$
$$\gamma(N) = M|S_N|^2 = M|\eta(N) - \eta(0)|^2 \leqslant 4M|\eta|^2 < \infty.$$

Thus, we demonstrated that either $\gamma(N) \to \infty$, or $\sup \gamma(N) < \infty$, the last/latter inequality occurring, only if $\xi(t)$ is represented in the form of the difference

$$\xi(t) = U^{t+1}\eta - U^{t}\eta = \eta(t+1) - \eta(t), \quad \eta \in H. \quad (3.7)$$

It is obvious, from (3.7) in turn, it follows that $\sup\limits_{N} \gamma(N) \leqslant 4M|\eta^2| < \infty$.

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It remained to check that (3.7) it is equivalent (3.5). Let it is made (3.7). Let us designate by $f_\eta(\lambda)$ the spectral density of sequence $\eta(t)$. On the strength of (3.7)

$$f_\eta(\lambda) = \frac{f(\lambda)}{|e^{i\lambda} - 1|^2} = \frac{f(\lambda)}{4\sin^2\frac{\lambda}{2}},$$

so that

$$\int_{-\pi}^{\pi} \frac{f(\lambda)}{\sin^2\frac{\lambda}{2}} d\lambda = 4 \int_{-\pi}^{\pi} f_\eta(\lambda)\, d\lambda < \infty.$$

On the contrary, if is final integral (3.5), then on the strength of the determination itself $\gamma(N)$

$$\gamma(N) \leqslant \int_{-\pi}^{\pi} \frac{f(\lambda)}{\sin^2\frac{\lambda}{2}} d\lambda < \infty.$$

Lemma is demonstrated.

Lemma 5. With each μ, when $N\to\infty$, function $\gamma(N, \mu)$ either approaches ∞ or is limited. The latter occurs in that and only that case, if

$$\int_{-\pi}^{\pi} \frac{f(\lambda)}{\sin^2 \frac{\lambda-\mu}{2}}\, d\lambda < \infty.$$

Proof. After presenting function $\gamma(N; \mu)$ in the form

$$\gamma(N;\mu) = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda+\mu)\, d\lambda,$$

we are convinced that $\gamma(N; \mu)$ it coincides with expression $M\left|\sum_{1}^{n} \xi'(t)\right|^2$, where the stationary in the broad sense sequence $\{\xi'(t)\}$ has as its spectral density $f(\lambda+\mu)$. Further follows from lemma 4.

Lemma 6. With $N\to\infty$ either $\inf_{\mu} \gamma(N, \mu)\to\infty$, or there is a point such, that $\theta\in[-\pi, \pi]$

$$\int_{-\pi}^{\pi} \frac{f(\lambda)}{\sin^2 \frac{\lambda-\theta}{2}}\, d\lambda < \infty.$$

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Proof differs little from the proof of lemma 4. Let $\varliminf_{N} \inf_{\mu} \gamma(N, \mu) < \infty$. Then it is possible to isolate sequence N_k and convergent series $\{\theta_k\}$ points from $[-\pi, \pi]$ with limit θ so that

$$\lim_{k} \gamma(N_k;\ \theta_k) < \infty.$$

By set/assuming $S_k = \sum_{0}^{N_k - 1} e^{it\theta_k} \xi(t)$ and by discussing as in the proof of lemma 4, let us arrive at the existence of maximum cell/element $\eta \in H$ such, that with all $\zeta \in H$

$$\lim_{k} \mathrm{M} S_k \zeta = \mathrm{M} \eta \zeta,$$
$$\lim_{k} \mathrm{M} e^{-i\theta_k} U S_k \zeta = e^{-i\theta} \mathrm{M} U S_k \zeta.$$

From these equalities just as it is above, is derived the representation

$$\xi(t) = \eta(t) - e^{-i\theta} \eta(t+1),$$

from which in turn, escape/ensues the assertion of lemma.

Lemma 7. Spectral density f (λ) of sequence $\{\xi(t)\}$, which satisfies the condition of full/total/complete regularity, is represented in the form

$$f(\lambda) = w(\lambda) \,|\, P(e^{i\lambda}) \,|^2, \tag{3.8}$$

where P (z) - polynomial with roots on $|z| = 1$, a w (λ) it possesses those by property, that

$$\varliminf_{N} \inf_{\mu} \int_{-\pi}^{\pi} \frac{\sin^2 N \frac{\lambda - \mu}{2}}{\sin^2 \frac{\lambda - \mu}{2}} w(\lambda)\, d\lambda = \infty.$$

Proof. Let us assume

$$\sigma^2(n) = \min \mathrm{M} \,|\, \xi(0) - \eta_n \,|^2, \qquad (3.9)$$

where min it is taken on all $\eta_n \in H\ (|t| > n)$.

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Let us designate η_n^* the random variable from $H\ (|t|>n)$, on which is reached the minimum in (3.9). It is obvious,

$$\mathrm{M}\,|\,\xi(0)\,|^2 + \mathrm{M}\,|\,\eta_n^*\,|^2 - 2\sqrt{2}\,\rho(n)\,\mathrm{M}^{1/2}\,|\,\xi(0)\,|^2\,\mathrm{M}^{1/2}\,|\,\eta_n^*\,|^2 \leqslant$$
$$\leqslant \sigma^2(n) \leqslant \mathrm{M}\,|\,\xi(0)\,|^2. \qquad (3.10)$$

Therefore with large n

$$\sigma^2(n) \geqslant \frac{1}{2}\,\mathrm{M}\,|\,\xi(0)\,|^2 > 0, \qquad (3.11)$$

i.e. the process ξ (t) is "noninterpolated", and hence follows [1] the existence of polynomial Q (z) such, that

$$\int_{-\pi}^{\pi} \frac{|\,Q(e^{i\lambda})\,|^2}{f(\lambda)}\,d\lambda < \infty. \qquad (3.12)$$

FOOTNOTE [1]. See, for example, [22], page 142. ENDFOOTNOTE.

Let us designate Q_0 (z) that of the polynomials Q (z), that satisfy (3.12), which has a smallest degree and coefficient 1 with leading term. Inequality (3.12) indicates also that among all polynomials Q (z) with the real roots, for which $\int_{-\pi}^{\pi} \frac{f(\lambda)}{|\,Q(e^{i\lambda})\,|^2}\,d\lambda < \infty$, there is polynomial P (z) of the maximum (final) degree. Actually,

from the inequality

$$\left(\int_{-\pi}^{\pi} \left| \frac{Q_0(e^{i\lambda})}{P(e^{i\lambda})} \right|^2 d\lambda \right)^2 \leqslant \int_{-\pi}^{\pi} \frac{|Q_0(e^{i\lambda})|^2}{f(e^{i\lambda})} d\lambda \int_{-\pi}^{\pi} \frac{f(e^{i\lambda})}{|P(e^{i\lambda})|^2} d\lambda < \infty$$

it follows that polynomial Q_0 is divided into P.

Let us assume now $f(\lambda) = |P(e^{i\lambda})|^2 w(\lambda)$ and let us assume that

$$\liminf_{N} \inf_{\mu} \int_{-\pi}^{\pi} \frac{\sin^2 N \frac{\lambda-\mu}{2}}{\sin^2 \frac{\lambda-\mu}{2}} w(\lambda)\, d\lambda < \infty.$$

Then on lemma 6 is located the point $\theta \in [-\pi, \pi]$, for which

$$\int_{-\pi}^{\pi} \frac{w(\lambda)}{|1-e^{i(\lambda-\theta)}|^2} d\lambda = \int_{-\pi}^{\pi} \frac{f(\lambda)\, d\lambda}{|P(e^{i\lambda})(1-e^{i(\lambda-\theta)})|^2} < \infty.$$

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The last/latter inequality, obviously, contradicts assumption about the maximality of polynomial P (z). Lemma is demonstrated.

Lemma 8. If f (λ) - the spectral density of the sequence $\{\xi(t)\}$, which satisfies the condition of full/total/complete regularity, then at those points μ, where $\lim \gamma(N;\mu) = \infty$, function γ (N; μ) is represented in the form γ (N; μ) = Nh (N; μ), where h (N; μ) is slowly varying

(in the sense of Karamat) function N, i.e., with all whole k > 0

$$\lim_{N\to\infty}\frac{h(kN;\mu)}{h(N;\mu)}=1.$$

Proof. Let us assume for the multiplicity

$$\gamma(N;\mu)=\gamma(N),\quad h(N;\mu)=h(N).$$

Let, further,

$$\begin{aligned} z_j=\sum_{s=1}^{N}\xi[(j-1)N+(j-1)r+s]\times \\ \times\exp\{-i\mu[(j-1)N+(j-1)r+s]\}, \\ j=1,\ \ldots,\ k; \\ y_j=\sum_{s=1}^{r}\xi[jN+(j-1)r+s]\exp\{-i\mu[jN+(j-1)r+s]\}, \\ j=1,\ \ldots,\ k-1; \\ y_k=-\sum_{s=1}^{(k-1)r}\xi(Nk+s)\exp\{-i\mu(Nk+s)\}, \end{aligned}\tag{3.13}$$

where r = r (N)→∞ with N→∞, but so be slow, in order to $\frac{\gamma(r)}{\gamma(N)}\to 0,\ \frac{\gamma((k-1)r)}{\gamma(N)}\to 0.$

Then

$$\begin{aligned}\gamma(N)=\mathsf{M}\,|z_1+y_1+\ \ldots\ +z_k+y_k|^2= \\ =\sum_{j=1}^{k}\mathsf{M}\,|z_j|^2+\sum_{i,j}(\mathsf{M}z_i\bar{y}_j+\mathsf{M}\bar{z}_iy_j)+\sum_{i\neq j}\mathsf{M}z_i\bar{z}_j+\sum_{i,j}\mathsf{M}y_i\bar{y}_j.\end{aligned}\tag{3.14}$$

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On the strength of stability the first sum to the right in (3.14) is equal to $k\gamma(N)$, the second and fourth sums in (3.14) will not exceed $k^2(\gamma(N)\gamma(r))^{1/2}=o(\gamma(N))$ finally the third sum in

(3.14) on the strength of the condition of full/total/complete regularity it will not exceed $k^2\gamma(N)\rho(r) = o(\gamma(N))$. Thus, $\gamma(kN) = k\gamma(N)(1 + o(1))$. Lemma is demonstrated.

Are below we will show that function h (N) can be continued from integers to all positive numbers with the preservation/retention/maintaining of property to be slowly varying. For the arbitrary slowly varying functions of natural argument this continuation, generally speaking, is impossible, and we first let us note those properties h (N), which make this continuation possible. During the conclusion/derivation of these properties we will set/assume $\mu = 0$; the case $\mu \neq 0$ is examined analogously, necessary only values $\xi(t)$ to everywhere replace by $\xi(t)e^{-i\mu t}$.

1. If k is fix/recorded and $N\to\infty$, then $\lim \frac{h(N+k)}{h(N)} = 1$.

Actually, since $\gamma(N)\to\infty$ with $N\to\infty$,

$$\gamma(N+k) = \gamma(N) + \gamma(k) + 2M(\xi(1) + \ldots + \xi(N))(\xi(N+1) + \ldots + \xi(N+k)), \quad (3.15)$$
$$|(\xi(1) + \ldots + \xi(N))(\xi(N+1) + \ldots + \xi(N+k))| \leqslant (\gamma(N)\gamma(k))^{1/2}$$

and

$$\frac{\gamma(N+k)}{\gamma(N)} = \frac{N+k}{N}\frac{h(N+k)}{h(N)} = 1 + o(1), \quad (3.16)$$
$$\frac{h(N+k)}{h(N)} = 1 + o(1).$$

2. With all $\varepsilon>0$ $\lim\limits_{N\to\infty} N^{\varepsilon}h(N)=\infty,\ \lim\limits_{N\to\infty} N^{-\varepsilon}h(N)=0.$

Actually, using relationship/ratio h (2N) $\sim$ h (N) and the previous property, we have

$$\ln h(N)=\sum_{l}\ln\frac{h\left(\left[\frac{N}{2^{l}}\right]\right)}{h\left(\left[\frac{N}{2^{l+1}}\right]\right)}=o(\ln N).$$

3. If r $\leqslant$ p $\leqslant$ 2r, then for sufficiently large r

$$\sup_{p}\frac{h(p)}{h(r)}\leqslant 4.$$

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Let us select and we record/fix m by the so/such large that ρ (m) < 1/16. Let us assume that p > 3r/2 (case p $\leqslant$ 3r/2 is examined analogously). From the representation

$$\sum_{1}^{p+m}\xi(t)=\sum_{1}^{r}\xi(t)+\sum_{r+1}^{r+m}\xi(t)+\sum_{r+m+1}^{p+m}\xi(t)$$

we find that

$$(p+m)\,h(p+m)=rh(r)+(p-r)\,h(p-r)+\theta,$$

where

$$|\theta|\leqslant 2\left[\rho(m)\left(r(p-r)\,h(r)\,h(p-r)\right)^{1/2}+\left(rmh(r)\,h(m)\right)^{1/2}+\right.$$
$$\left.+\left((p-r)\,mh(m)\,h(p-r)\right)^{1/2}\right]+mh(m).$$

It is obvious,

$$2\left(r(p-r)\,h(r)\,h(p-r)\right)^{1/2}\leqslant rh(r)+(p-r)\,h(p-r).$$

Therefore with large r $h(p+m)=\theta_1 h(r)+\theta_2 h(p-r)+R_p$, where $R_p=O(p^{-1/4})$, $\theta_1>15/32$, $\theta_2>0$, and therefore with large r (see the previous property)

$$\theta_1\frac{h(r)}{h(r+m)}<\frac{3}{2},\quad \theta_1\frac{h(r)}{h(p)}<\frac{4}{3},$$
$$\frac{h(r)}{h(p)}<4.$$

4. For all sufficiently small c and sufficiently large N $h(cN)/h(N)<c^{-1/2}$.

It goes without saying that they are accepted into consideration only those N, for which cN - whole. We have, using properties 1 and 2,

$$\ln \frac{h(cN)}{h(N)} \sum_{k=0}^{\left[-\frac{\ln c}{\ln 2}\right]} \left(\ln h\left(\left[\frac{N}{2^{k+1}}\right]\right) - \ln h\left(\left[\frac{N}{2^k}\right]\right)\right) +$$
$$+ \ln h(cN) - \ln h\left(\left[\frac{N}{2^{\left[-\frac{\ln c}{\ln 2}\right]}}\right]\right) < \frac{1}{2} \ln|c| = -\frac{1}{2}\ln c. \quad (3.17)$$

Let us note that the demonstrated inequality occurs evenly on all sufficiently small c, $c < c_0$.

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Let us spread now functions γ (N), h (N) to all positive x, by set/assuming

$$\gamma(x;\ \mu) = \int_{-\pi}^{\pi} \frac{\sin^2 \frac{x\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda + \mu)\, d\lambda,$$
$$h(x;\ \mu) = \frac{1}{x} \gamma(x,\ \mu). \quad (3.18)$$

Lemma 9. With all μ, for which γ (N)$\to\infty$, function h (x) - slowly varying, i.e., with all $y > 0$

$$\lim_{x\to\infty} \frac{h(xy)}{h(x)} = 1. \quad (3.19)$$

Proof. We will demonstrate (3.19) first for rational y. It is obvious, with $x \to \infty$

$$\gamma(x) = \gamma([x])(1 + o(1)). \quad (3.20)$$

Therefore with y = k, where k is integer, on the strength of the property of 1 functions h (N)

$$\frac{\gamma(kx)}{\gamma(x)} = \frac{[kx]}{[x]} \frac{h([kx])}{h([t])} (1 + o(1)) = k(1 + o(1)). \quad (3.21)$$

If y = p/q, where p, q are whole, then, by taking into account (3.21), let us find

$$\lim_{x \to \infty} \frac{\gamma\left(\frac{p}{q} x\right)}{\gamma(x)} = \lim \frac{\gamma\left(p \frac{x}{q}\right)}{\gamma\left(\frac{x}{q}\right)} \frac{\gamma\left(\frac{x}{q}\right)}{\gamma\left(q \frac{x}{q}\right)} = \frac{p}{q}. \quad (3.22)$$

Let now y be an arbitrary real number. Let us assume

$$\psi_1(y) = \varliminf_{x \to \infty} \frac{\gamma(x, y)}{\gamma(x)}, \quad \psi_2(y) = \varlimsup_{x \to \infty} \frac{\gamma(x, y)}{\gamma(x)}.$$

For rational y $\psi_1(y) = \psi_2(y) = y$, and therefore to us is sufficient to show that the functions $\psi_1(y)$, $\psi_2(y)$ are continuous.

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Since

Since

$$\frac{|\gamma((y+\varepsilon)x)-\gamma(yx)|}{\gamma(x)} \leqslant \frac{1}{\gamma(x)}\int_{-\pi}^{\pi}\frac{\sin^2\frac{\varepsilon x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,d\lambda + $$
$$+\frac{1}{2\gamma(x)}\left|\int_{-\pi}^{\pi}\frac{\sin\varepsilon x\lambda\,\sin yx\lambda}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,d\lambda\right| \leqslant$$
$$\leqslant\frac{\gamma(\varepsilon x)}{\gamma(x)}+\frac{1}{2}\left(\frac{\gamma(yx)}{\gamma(x)}\right)^{1/2}\left(\frac{\gamma(\varepsilon x)}{\gamma(x)}\right)^{1/2}, \quad (3.23)$$

it suffices to demonstrate continuity $\psi_1(y)$ and $\psi_2(y)$ in zero. Using a property of 3 functions h (N), we have

$$\frac{\gamma(\varepsilon x)}{\gamma(x)}=\frac{[\varepsilon x]}{[x]}\frac{h\left(\frac{[\varepsilon x]}{[x]}[x]\right)}{h([x])}(1+o(1))\leqslant\varepsilon^{1/2}(1+o(1)). \quad (3.24)$$

From (3.23), (3.24) it follows that

$$|\psi_1(y+\varepsilon)-\psi_1(y)|=O(\varepsilon^{1/4}), \quad |\psi_2(y+\varepsilon)-\psi_2(y)|=O(\varepsilon^{1/4}).$$

Lemma is demonstrated.

Lemma 10. If $\gamma_0(N)=\inf_{\mu}\gamma(N;\mu)\xrightarrow[N\to\infty]{}\infty$, then the relationship/ratio

$$\lim_{x\to\infty}\frac{h(xy)}{h(x)}=1$$

occurs evenly on all μ and y as such, that $0 < y_0 < y < y_1 < \infty$.

Proof. Let $\frac{h(xy)}{h(x)}=1+R(\mu;y;x)$. Let first $x = N$, $y = k$ be integers. As it follows from the proof of lemma 8,

$$R(\mu;k;N)=O\left[k\rho(r)+k\left(\frac{\gamma(r;\mu)}{\gamma_0(N)}\right)^{1/2}+k\left(\frac{\gamma((k-1)r;\mu)}{\gamma_0(N)}\right)^{1/2}\right].$$

With all μ $\gamma(N;\mu)<N^2\int_{-\pi}^{\pi} f(\lambda)\,d\lambda$, and therefore, by set/assuming $r=\ln\gamma_0(N)$, we will obtain that

$$R(\mu;\ k;\ N)=O\left[k\,\rho(r)+\left(\frac{\ln^2\gamma_0(N)}{\gamma_0(N)}\right)^{1/2}\right].$$

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Further, from the proof of relationship/ratio (3.16) is concluded, that

$$\frac{\gamma(N+k;\ \mu)}{\gamma(N;\ \mu)}=1+O\left(\frac{k^2}{\gamma_0(N)}\right).$$

Hence it follows that it is evenly on μ

$$\ln h(N;\ \mu)=\sum\left(\ln h\left(\left[\frac{N}{2^{k-1}}\right];\ \mu\right)-\ln h\left(\left[\frac{N}{2^k}\right];\ \mu\right)\right)+ \\ +O(1)=o(\ln N),$$

and, therefore, $\lim\limits_{N} h(N;\ \mu)N^{-\varepsilon}=0$ evenly on μ. From entire aforesaid above already easily it is obtained, that property 3 and 4 functions h (N; μ) and equalities (3.20), (3.24) also occur evenly on μ.

Let now $y\in[y_0,\ y_1]$. Let us assign integer q. Is evenly on μ and p by such, that $y_0 < p/q < y_1$.

$$\lim_x \frac{h(px/q)}{h(x)}=1.$$

Let y' be nearest to y the number of form p/q. Then from (3.23), (3.24) it follows that

$$\left|\frac{\gamma(yx)}{\gamma(x)}-y\right| \leqslant \left|\frac{\gamma(y'x)}{\gamma(x)}-y'\right| + |y'-y| + \\ + \left|\frac{\gamma(yx)}{\gamma(x)}-\frac{\gamma(y'x)}{\gamma(x)}\right| \leqslant Cq^{-1/4}+o(1),$$

where constant C and o (1) do not depend on μ, q. Lemma is demonstrated.

§4. Local conditions (termination).

Now we pass directly to the proof of theorem 5. This theorem can be considered as proposition is Tauberian the type, in which on the basis of the behavior of the function

$$\gamma(N;\mu)=\int_{-\pi}^{\pi}\frac{\sin^2 N\frac{\lambda-\mu}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda)\,d\lambda \quad \textbf{with} \quad N\to\infty$$

are made the conclusions relative to the behavior

$$\frac{1}{x}\int_0^x f(\lambda+\mu)\,d\lambda \quad \text{with} \quad x\to 0.$$

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In the proof of this theorem will be used the belonging to Karamat the method of the proof of Tauberian theorems.

Let us show preliminarily that with all μ we can count function γ (N; μ) representable in the form

$$\gamma(N;\mu)=N\cdot h(N;\mu),$$

where h (N; μ) is the slowly varying function. This follows of lemma 7 and given below lemma 11.

Lemma 11. If f (λ) - the spectral density of the sequence $\{\xi(t)\}$, which satisfies the condition of full/total/complete regularity, all zeros of the polynomial P (z) are equal on module/modulus 1, and function $w(\lambda)=\frac{f(\lambda)}{|P(e^{i\lambda})|^2}$ is integrated on $[-\pi,\pi]$, then w (λ) is the spectral density of the sequence $\{\eta(t)\}$, which also satisfies the condition of full/total/complete regularity, and ρ (τ; w) $\leqslant$ ρ (τ; f).

Proof. Let function $\varphi,\ \psi\in\mathscr{H}^2$. If polynomial P (z) has roots only

on circumference $|z|=1$, then for all sufficiently small δ $\varphi/P \in \mathcal{H}^{\delta}$, $\psi/\bar{P} \in \mathcal{H}^{\delta}$.

Therefore equality (1.3) gives

$$\rho(\tau;\ w) = \sup_{\varphi,\ \psi} \left| \int_{-\tau}^{\tau} \varphi(\lambda)\, \psi(\lambda)\, e^{i\lambda\tau} w(\lambda)\, d\lambda \right| =$$
$$= \sup_{\varphi,\ \psi} \left| \int_{-\tau}^{\tau} \frac{\varphi(\lambda)}{P(\lambda)} \frac{\psi(\lambda)}{\bar{P}(\lambda)} e^{i\lambda\tau} w(\lambda) \,|\, P(e^{i\lambda})\,|^2\, d\lambda \right| \leqslant$$
$$\leqslant \sup_{\varphi,\ \psi} \left| \int_{-\tau}^{\tau} \varphi(\lambda)\, \psi(\lambda)\, e^{i\lambda\tau} f(\lambda)\, d\lambda \right| = \rho(\tau;\ f).$$

Here in the first integral sup it is taken on all $\varphi, \psi \in L^{+}(w)$ with $\|\varphi\|_w = \|\psi\|_w = 1$, and in the latter - on all $\varphi, \psi \in L^{+}(f)$ with $\|\varphi\|_f = \|\psi\|_f = 1$. Lemma is demonstrated.

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By beginning from this place and by ending with the proof of lemma 4, let us consider that through f (λ) is designated the spectral density of the stationary sequence $\{\xi(t)\}$, which satisfies the condition of full/total/complete regularity with the coefficient of regularity ρ (τ). Furthermore, let us assume that f (λ) is such, that for it $\lim_{N} \inf_{\mu} \gamma(N;\ \mu) = \infty$.

Lemma 12. Let a (λ) - the even function with the limited third derivative, which turns into zero outside interval $[-1, 1]$. Then with

x →it is evenly on μ

$$\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,a(x\lambda)\,d\lambda=$$

$$=\frac{1}{\pi}\int_0^1\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}a(\lambda)\,d\lambda\cdot h(x)(1+o(1)).\quad(4.1)$$

Proof. Let $|a'''(\lambda)|\leqslant C<\infty$. Let, further,

$$a(\lambda)=\sqrt{\frac{2}{\pi}}\int_0^\infty\cos\lambda z A(z)\,dz,$$
$$A(z)=\sqrt{\frac{2}{\pi}}\int_0^\infty\cos\lambda z a(\lambda)\,d\lambda.\quad(4.2)$$

Based on smoothness $a(\lambda)$ $|A(z)|\leqslant\frac{C}{|z|^3+1}$. According to the assigned number $\varepsilon>0$ let us select numbers s and δ so, in order to

$$\int_s^\infty|A(z)|\,dz+\int_{|1-z|\leqslant\delta}|A(z)|\,dz+\int_0^\delta|A(z)|\,dz<\varepsilon.\quad(4.3)$$

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Let us assume $B=\{z\colon \delta\leqslant z\leqslant s,\ |1-z|>\delta\}$. Then

$$\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,a(x\lambda)\,d\lambda=$$
$$=\frac{1}{x}\sqrt{\frac{2}{\pi}}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,d\lambda\int_{0}^{\infty}\cos x\lambda z\cdot A(z)\,dz=$$
$$=\sqrt{\frac{2}{\pi}}\int_{B}\cdot A(z)\,dz\,\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}\cos x\lambda z\cdot f(\lambda+\mu)\,d\lambda+R(x),\tag{4.4}$$

where $|R(x)|\leqslant \varepsilon h(x)$. Using the identity

$$\cos x\lambda z\sin^2\frac{x\lambda}{2}=$$
$$=\frac{1}{2}\left[\sin^2\frac{x\lambda}{2}(z+1)+\sin^2\frac{x\lambda}{2}(z-1)-2\sin^2\frac{x\lambda}{2}z\right],$$

let us rewrite the right side of equality (4.4) in the form

$$\sqrt{\frac{1}{2\pi}}\Bigg[\int_{1+\delta}^{s} z\,[h(x(z+1))+h(x(z-1))-2h(xz)]\,A(z)\,dz+$$
$$+\int_{1+\delta}^{s}[h(x(z+1))-h(x(z-1))]\,A(z)\,dz+$$
$$+\int_{\delta}^{1-\delta} z\,[h(x(z+1))-h(x(1-z))-2h(xz)]\,A(z)\,dz+$$
$$+\int_{\delta}^{1-\delta}[h(x(1+z))+h(x(1-z))]\,A(z)\,dz\Bigg]+R(x). \quad (4.5)$$

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Taking into account that $\lim\frac{h(xz)}{h(z)}=1$ it is evenly from $\mu\in[-\pi,\pi]$, $\delta\leqslant z\leqslant s+1$ (lemma 10), we will obtain that the first and second integrals in (4.5) it is evenly in μ essence o (h (x)), and the sum of remaining terms, besides R (x), with $x\to\infty$ evenly on μ is converted in

$$\sqrt{\frac{2}{\pi}}\int_{\delta}^{1-\delta}(1-z)\,A(z)\,dz\cdot h(x)(1+o(1)).$$

Taking into account arbitrariness δ, ε, we find hence that it is evenly on μ

$$\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,a(x\lambda)\,d\lambda=$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1(1-z)\,A(z)\,dz\cdot h(x)(1+o(1))=$$

$$=\frac{2}{\pi}\int_0^1 a(\lambda)\,d\lambda\int_0^1(1-z)\cos\lambda z\,dz\cdot h(x)(1+o(1))=$$

$$=\frac{1}{\pi}\int_0^1\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}a(\lambda)\,d\lambda\cdot h(x)(1+o(1)). \qquad (4.6)$$

Lemma 13. Let a (λ) - the odd, three times differentiated function with the limited third derivative, which turns into zero outside interval $[-1, 1]$. Then with $x\to\infty$ it is evenly on μ

$$\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,a(x\lambda)\,d\lambda=o(h(x)). \qquad (4.7)$$

Proof. First let us assume that $x\to\infty$, by passing only integers, and instead of x let us write N.

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As it is above, let us introduce Fourier transform function a (λ):

$$A(z) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin \lambda z a(\lambda)\, d\lambda,$$
$$a(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin \lambda z\, A(z)\, dz. \quad (4.8)$$

According to that which was assigned $\varepsilon > 0$ let us determine the set

$$B_1 = \{z \geqslant s\} \cup \{z : |z - n| \leqslant \delta, \quad n = 0, 1, \ldots\}$$

so, in order to $\int_{B_1} |A(z)|\, dz < \varepsilon$. Let us designate B complement of a set B_1 to half-line [0, ∞]. Then

$$\frac{1}{N} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda + \mu)\, a(N\lambda)\, d\lambda =$$
$$= \frac{1}{N} \int_B A(z)\, dz \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N\lambda}{2}}{\sin^2 \frac{\lambda}{2}} \sin N\lambda z f(\lambda + \mu)\, d\lambda + R(N), \quad (4.9)$$
$$|R(N)| \leqslant \varepsilon h(N).$$

With whole N $\frac{\sin^2 \frac{N\lambda}{2}}{\sin^2 \frac{\lambda}{2}} = \left| \sum_0^{N-1} e^{il\lambda} \right|^2$. By decompose/expanding sin Nλz

according to degrees $e^{i\lambda}$, let us find that

$$\sin N\lambda z = \frac{\sin Nz\pi}{2\pi i} \sum_{j=1}^{\infty} (-1)^j (e^{ij\lambda} - e^{-ij\lambda}) \left(\frac{1}{Nz-j} - \frac{1}{Nz+j} \right) = \\ = \sum_{j=1}^{\infty} a_j(z)(e^{ij\lambda} - e^{-ij\lambda}). \quad (4.10)$$

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Now we can rewrite first term in right side (4.9) in the form

$$\int_B A(z)\,dz \frac{1}{N} \int_{-\pi}^{\pi} [\Phi_1(\lambda;\ z) + \Phi_2(\lambda;\ z)] f(\lambda+\mu)\,d\lambda - \\ - \frac{1}{N} \int_B A(z)\,dz \int_{-\pi}^{\pi} [\Phi_1(-\lambda;\ z) + \Phi_2(-\lambda;\ z)] f(\lambda+\mu)\,d\lambda + \\ + \frac{1}{N} \int_B A(z)\,dz \int_{-\pi}^{\pi} \Phi_0(\lambda;\ z) f(\lambda+\mu)\,d\lambda,$$

where

$$\Phi_1(\lambda;\ z) = \sum_{k=0}^{N-1} e^{ik\lambda} \sum_{j=1}^{N-1} a_j(z) \sum_{m=1}^{j} e^{im\lambda}, \\ \Phi_2(\lambda;\ z) = \sum_{j=1}^{N-1} a_j(z) \sum_{k=N-j}^{N-1} e^{ik\lambda} \sum_{m=0}^{N-j-1} e^{-imk}, \quad (4.11) \\ \Phi_0(\lambda;\ z) = 2i \frac{\sin^2 \frac{N\lambda}{2}}{\sin^2 \frac{\lambda}{2}} \sum_{j=N}^{\infty} a_j(z) \sin j\lambda.$$

Let us pass to the estimation of the integrals

$$\frac{1}{N}\int_{-\pi}^{\pi}\Phi_{l}(\lambda;\ z)f(\lambda+\mu)\,d\lambda.$$

1. Estimation $\frac{1}{N}\int_{-\pi}^{\pi}\Phi_{0}(\lambda;\ z)f(\lambda+\mu)\,d\lambda.$ Let us note that $\left|\sum(-1)^{k}\sin k\lambda\right| \leqslant C$ with all n, where C is an absolute constant. By using the transform of Abel, we will obtain

$$\sum_{j=N}^{\infty}a_{j}(z)\sin j\lambda=\frac{\sin Nz\pi}{2\pi}\sum_{j=N}^{\infty}\Big[\Big(\frac{1}{Nz-j}-\frac{1}{Nz-j-1}\Big)-$$

$$-\Big(\frac{1}{Nz+j}-\frac{1}{Nz+j+1}\Big)\Big]\sum_{k=N}^{j}(-1)^{k}\sin k\lambda. \qquad (4.12)$$

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From (4.12) easily it follows that with z < 1 - δ

$$|\Phi_0(\lambda;\ z)| \leqslant \frac{C}{\delta N}. \tag{4.13}$$

Let now z > 1 + δ. Let us designate n the large, but fixed/recorded number. Again by using the transform of Abel, let us find that for sufficiently large N

$$\left|\sum_{|j-Nz|>n} a_j(z)\sin j\lambda\right| \leqslant C\left(\frac{1}{N\delta}+\frac{1}{n}\right). \tag{4.14}$$

From the condition of full/total/complete regularity it follows that

$$\begin{aligned}
&\left|\frac{1}{N}\int_{-\pi}^{\pi}\frac{\sin^2\frac{N\lambda}{2}}{\sin^2\frac{\lambda}{2}}\sum_{|j-Nz|\leqslant n} a_j(z)\left(e^{ij\lambda}-e^{-ij\lambda}\right) f(\lambda+\mu)\,d\lambda\right| = \\
&=\left|\sum_{|j-Nz|\leqslant n}\frac{a_j(z)}{N}\int_{-\pi}^{\pi}\left[\left(\sum_{0}^{N-1} e^{ik(\lambda-\mu)}\right)^2 e^{i(j-N+1)(\lambda-\mu)}+\right.\right. \\
&\left.\left.+\left(\sum_{0}^{N-1} e^{-ik(\lambda-\mu)}\right)^2 e^{-i(j-N+1)(\lambda-\mu)}\right] f(\lambda)\,d\lambda\right| \leqslant \\
&\leqslant \frac{4n}{N}\rho(N\delta-n)\int_{-\pi}^{\pi}\frac{\sin^2\frac{N\lambda}{2}}{\sin^2\frac{\lambda}{2}} f(\lambda+\mu)\,d\lambda = o(h(N)).
\end{aligned} \tag{4.15}$$

Finally (4.13) - (4.15) they give, which is evenly on μ

$$\frac{1}{N}\left|\int_{B} A(z)\,dz \int_{-\pi}^{\pi} \Phi_0(\lambda;\ z) f(\lambda+\mu)\,d\lambda\right| = o(h(N)). \quad (4.16)$$

2. Estimation $\frac{1}{N}\int_{-\pi}^{\pi}\Phi_1(\lambda;\ z) f(\lambda+\mu)\,d\lambda.$ Let us designate again by n the sufficiently large, but fixed/recorded number. Let us decompose the external sum of formula for $\Phi_1(\lambda;\ z)$ (see (4.11) to two sums: $\sum\limits_{k=0}^{n}$ and $\sum\limits_{k=n+1}^{N-1}$.

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Integral of the first sum will not exceed

$$\frac{n}{N}\int_{-\pi}^{\pi}\left|\sum_{l=1}^{N-1} a_l(z)\sum_{m=0}^{l-1} e^{im\lambda}\right| f(\lambda+\mu)\,d\lambda =$$
$$= \frac{n}{2N}\int_{-\pi}^{\pi}\frac{1}{\left|\sin\frac{\lambda}{2}\right|}\left|\sum_{1}^{N} a_l(z)(e^{il\lambda}-1)\right| f(\lambda+\mu)\,d\lambda \leqslant$$
$$\leqslant \frac{Cn}{N}\int_{-\pi}^{\pi}\frac{1}{\left|\sin\frac{\lambda}{2}\right|}\left|\sum_{l=1}^{N-1} a_l(z)\sin l\lambda\right| f(\lambda+\mu)\,d\lambda +$$
$$+ \frac{Cn}{N}\int_{-\pi}^{\pi}\frac{1}{\left|\sin\frac{\lambda}{2}\right|}\left|\sum_{l=1}^{N} a_l(z)\sin^2\frac{l\lambda}{2}\right| f(\lambda+\mu)\,d\lambda. \quad (4.17)$$

With the help of relationship/ratios (4.10) and (4.13) - (4.15) easily we find that right side (4.17) evenly on μ does not exceed

$$\frac{Cn}{N}\int_{-\pi}^{\pi}\left|\frac{\sin N\lambda z}{\sin\frac{\lambda}{2}}\right| f(\lambda+\mu)\,d\lambda + o(h(N)) \leqslant C\,\frac{n\sqrt{z}}{\sqrt{N}}\,h(Nz) +$$
$$+ o(h(N)). \quad (4.18)$$

For the second of the component, components

$$\int_{-\pi}^{\pi} \Phi_1(\lambda;\ z) f(\lambda+\mu)\, d\lambda,$$

we have, using a transform of Abel,

$$\frac{1}{N}\int_{-\pi}^{\pi}\left(\sum_{k=n}^{N-1} e^{ik\lambda}\sum_{j=1}^{N-1} a_j(z)\sum_{1}^{j} e^{im\lambda}\right) f(\lambda+\mu)\, d\lambda =$$

$$= -\frac{1}{N}\int_{-\pi}^{\pi} e^{in\lambda}\left(\sum_{k=0}^{N-n-1} e^{ik\lambda}\sum_{j=1}^{N-2}\left(\sum_{s=1}^{j} a_s(z)\right) e^{i(j+1)\lambda}\right) f(\lambda+\mu)\, d\lambda +$$

$$+\frac{1}{N}\int_{-\pi}^{\pi} e^{in\lambda}\left(\sum_{k=0}^{N-n-1} e^{ik\lambda}\sum_{m=1}^{N-1} e^{im\lambda}\sum_{j=1}^{N-1} a_j(z)\right) f(\lambda+\mu)\, d\lambda. \quad (4.19)$$

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Let us note that with $z \leqslant 1-\delta$, $j \leqslant Nz$

$$\sum_{s=1}^{j} a_s(z) = \frac{1}{2\pi i}\sum_{s=1}^{j}\frac{(-1)^s \sin Nz\pi}{Nz-s} + O\left(\frac{1}{N\delta}\right) =$$

$$= O\left(\frac{1}{N\delta} + \frac{|\sin Nz\pi|}{|Nz-j|}\right); \quad (4.20)$$

with $z \leqslant 1-\delta,\ j > Nz$

$$\sum_{s=1}^{j} a_s(z) = \frac{1}{2\pi i}\sum_{-\infty}^{\infty}\frac{(-1)^s \sin Nz\pi}{Nz-s} + O\left(\frac{1}{N\delta} + \frac{1}{|Nz-j|}\right) \quad (4.21)$$

and finally, when z > 1 + δ

$$\sum_{s=1}^{l} a_s(z) = O\left(\frac{1}{N\delta}\right). \qquad (4.22)$$

Let z ⩽ 1 - δ; then, by using (4.20), (4.21) and the condition of full/total/complete regularity, we will obtain for the module/modulus of right side (4.19) the following upper limit:

$$\frac{C}{N}\left(\frac{1}{\delta}+\ln N\right)\int_{-\pi}^{\pi}\left|\frac{\sin\frac{N-n}{2}\lambda}{\sin\frac{\lambda}{2}}\right| f(\lambda+\mu)\,d\lambda +$$
$$+\frac{C\,\theta(z)}{N}\,\rho(n+Nz)\left(\int_{-\pi}^{\pi}\frac{\sin^2\frac{N-n}{2}\lambda}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,d\lambda\right)^{1/2}\times$$
$$\times\left(\int_{-\pi}^{\pi}\frac{\sin^2 N\frac{1-z}{2}\lambda}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,d\lambda\right)^{1/2} = o(h(N)), \qquad (4.23)$$

where

$$\theta(z) = \frac{1}{2\pi i}\sum_{-\infty}^{\infty}\frac{(-1)^s \sin Nz\pi}{Nz-s}.$$

Similarly with z > 1 + δ we obtain as upper limit for (4.19) the expression

$$\frac{C}{N\delta}\int_{-\pi}^{\pi}\left|\frac{\sin\frac{N-n}{2}\lambda}{\sin\frac{\lambda}{2}}\right| f(\lambda+\mu)\,d\lambda = o(h(N)). \qquad (4.24)$$

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It is obvious, and (4.23) and (4.24) they occur evenly on μ and $z \in B$. From (4.18) and (4.24) we obtain, which is evenly on μ and $z \in B$

$$\frac{1}{N}\left|\int_{-\pi}^{\pi}\Phi_1(\lambda;\ z)\,f(\lambda+\mu)\,d\lambda\right| = o(h(N)).$$

3. Estimation

$$\int_{-\pi}^{\pi}\Phi_2(\lambda;\ z)\,f(\lambda+\mu)\,d\lambda.$$

We have, record/fixing as before certain large number n,

$$\begin{aligned}
\frac{1}{N}\left|\int_{-\pi}^{\pi}\Phi_2(\lambda;\ z)\,f(\lambda+\mu)\,d\lambda\right| &= \\
&= \frac{1}{N}\left|\int_{-\pi}^{\pi}\left(\sum_{l=1}^{N-1}a_l(z)\sum_{k=0}^{N-1}e^{ik\lambda}\sum_{m=1}^{N-l}e^{im\lambda}\right)f(\lambda+\mu)\,d\lambda\right| \leqslant \\
&\leqslant \frac{1}{N}\left|\int_{-\pi}^{\pi}\left(\sum_{l=1}^{n}a_l(z)\sum_{k=0}^{l-1}e^{ik\lambda}\sum_{m=1}^{N-l}e^{im\lambda}\right)f(\lambda+\mu)\,d\lambda\right| + \\
&+ \frac{1}{N}\left|\int_{-\pi}^{\pi}\left(\sum_{l=n+1}^{N-1}a_l(z)\sum_{m=1}^{N-l}e^{im\lambda}\right)\sum_{k=0}^{n}e^{ik\lambda}f(\lambda+\mu)\,d\lambda\right| + \\
+\frac{1}{N}\left|\int_{-\pi}^{\pi}e^{in\lambda}\left(\sum_{l=n+1}^{N-1}a_l(z)\sum_{k=1}^{l-n}e^{ik\lambda}\sum_{m=1}^{N-l}e^{im\lambda}\right)f(\lambda+\mu)\,d\lambda\right|&. \quad (4.25)
\end{aligned}$$

It is obvious, first term by right side (4.25) will not exceed

$$\begin{aligned}
\frac{n}{N}\sum_{l=1}^{n}|a_l(z)|\int_{-\pi}^{\pi}\left|\frac{\sin\frac{N-l}{2}\lambda}{\sin\frac{\lambda}{2}}\right|f(\lambda+\mu)\,d\lambda &= \\
&= O\left(\frac{h(N)}{\sqrt{N}}\right) = o(h(N)). \quad (4.26)
\end{aligned}$$

By applying to second and third terms in right side (4.25) the transform of Abel and by discussing just as during the conclusion of estimations for (4.19), let us find that both these component is evenly in μ and $z \in B$ essence o (h (N)). Thereby

$$\frac{1}{N}\left|\int_{-\pi}^{\pi} \Phi_2(z;\ \lambda) f(\lambda+\mu)\, d\lambda\right| = o(h(N)). \qquad (4.27)$$

Thus, for whole x lemma is demonstrated. Let x be an arbitrary real number. Above we saw (see §3) that is evenly on μ h (x) = h ([x]) (1 + o (1)). Therefore it is evenly on μ

$$\left|\frac{1}{x}\int_{-\pi}^{\pi} \frac{\sin^2 \frac{x\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda+\mu)\, a([x]\lambda)\, d\lambda - \right.$$
$$\left. - \frac{1}{[x]}\int_{-\pi}^{\pi} \frac{\sin^2 \frac{[x]\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda+\mu)\, a([x]\lambda)\, d\lambda\right| \leqslant Cx^{-1/4} = o(h(x)). \qquad (4.28)$$

Finally,

Finally

$$\left| \frac{1}{x} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{x\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda+\mu)\left(a([x]\lambda) - a(x\lambda)\right) d\lambda \right| \leqslant$$

$$\leqslant \left| \frac{\max\limits_{\lambda} |a'(\lambda)|}{x} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{x\lambda}{2}}{\sin^2 \frac{\lambda}{2}} |\lambda| f(\lambda+\mu)\, d\lambda \right| \leqslant$$

$$\leqslant C\left(\frac{h(x)}{x}\right)^{1/2} = o(h(x)). \quad (4.29)$$

Lemma 13 is demonstrated.

Lemma 14. Let a (λ) - the turning into zero outside interval [-1, 1] function of bounded variation.

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Then with $x \to \infty$ it is evenly on μ

$$\frac{1}{x} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{x\lambda}{2}}{\sin^2 \frac{\lambda}{2}} f(\lambda+\mu)\, a(x\lambda)\, d\lambda =$$

$$= \frac{h(x)}{2\pi} \int_{-1}^{1} \frac{\sin^2 \frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2} a(\lambda)\, d\lambda + o(h(x)). \quad (4.30)$$

Proof. Let first function a (λ) have the limited third derivative. By applying to even function $\frac{a(\lambda)+a(-\lambda)}{2}$ lemma 12, and to odd function $\frac{a(\lambda)-a(-\lambda)}{2}$ lemma 13, we will obtain

$$\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,a(x\lambda)\,d\lambda=$$

$$=\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\frac{a(x\lambda)+a(-x\lambda)}{2}\,d\lambda+$$

$$+\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\frac{a(x\lambda)-a(-x\lambda)}{2}\,d\lambda=$$

$$=\frac{h(x)}{\pi}\int_{0}^{1}\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}\frac{a(\lambda)+a(-\lambda)}{2}\,d\lambda+o(h(x))=$$

$$=\frac{h(x)}{2\pi}\int_{-1}^{1}\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}a(\lambda)\,d\lambda+o(h(x)). \qquad (4.31)$$

Let now a (λ) have bounded variation. On any $\varepsilon>0$ always it is possible to select two satisfying the conditions of the lemma of function $\underline{a}(\lambda)$ and $\bar{a}(\lambda)$, which have the limited third derivative and such, that

$$\underline{a}(\lambda)\leqslant a(\lambda)\leqslant\bar{a}(\lambda),$$
$$\int_{-\pi}^{\pi}(\bar{a}(\lambda)-\underline{a}(\lambda))\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}\,d\lambda<\varepsilon. \qquad (4.32)$$

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Actually, for functions a (λ), the having only finite number jumps, this is obvious; but if the number of jumps a (λ) is infinite, we can preliminarily select two functions $\underline{b}(\lambda)$ and $\bar{b}(\lambda)$, that satisfy inequality of type (4.32) and having the finite number of jumps.

If functions $\underline{a}(\lambda)$ and $\bar{a}(\lambda)$ are selected in accordance with inequalities (4.32), then

$$\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\underline{a}(x\lambda)\,d\lambda\leqslant$$
$$\leqslant\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,a(x\lambda)\,d\lambda\leqslant$$
$$\leqslant\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,\bar{a}(x\lambda)\,d\lambda. \qquad (4.33)$$

For functions $\underline{a}(\lambda)$ and $\bar{a}(\lambda)$ it occurs (4.30), and therefore (4.33) it is possible to rewrite in the form

$$\frac{h(x)}{2\pi}\int_{-\pi}^{\pi}\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}\underline{a}(\lambda)\,d\lambda\,(1+o(1))\leqslant$$
$$\leqslant\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}f(\lambda+\mu)\,a(x\lambda)\,d\lambda\leqslant$$
$$\leqslant\frac{h(x)}{2\pi}\int_{-\pi}^{\pi}\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}\bar{a}(\lambda)\,d\lambda\,(1+o(1)). \qquad (4.34)$$

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In (4.34) and the right and left of part they differ from each other and from $\frac{h(x)}{2\pi}\int_{-\pi}^{\pi}\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}a(\lambda)\,d\lambda$ not more than on $\varepsilon h(x)(1+o(1))$, and, since ε arbitrarily, it is possible to claim that

$$\frac{1}{x}\int_{-\pi}^{\pi}\frac{\sin^2\frac{x\lambda}{2}}{\sin^2\frac{\lambda}{2}}f(\lambda+\mu)\,a(x\lambda)\,d\lambda = \frac{h(x)}{2\pi}\int_{-1}^{1}\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}a(\lambda)\,d\lambda\,(1+o(1)).$$

Lemma 14 is demonstrated. (Strictly speaking, still remained to lead further research of functions with discontinuity/interruptions in points ± 1; we will let to its reader).

Let us pass to the termination of the proof of theorem 5.

Let f (λ) - the spectral density of the Gaussian sequence ξ (t), which satisfies the condition of full/total/complete regularity. With the help of lemma 7 we can find the maximum polynomial P (z), for which the function $w(\lambda)=\frac{f(\lambda)}{|P(e^{i\lambda})|^2}$ is integrated and

$$\inf_{\mu} \int_{-\pi}^{\pi} \frac{\sin^2 \frac{N\lambda}{2}}{\sin^2 \frac{\lambda}{2}} w(\lambda+\mu)\, d\lambda \to \infty.$$

On lemma 11 w (λ) - the spectral density of the sequence, which satisfies the conlition of full/total/complete regularity. Let us use to w (λ) lemma 14, by set/assuming $a(\lambda) = \frac{\left(\frac{\lambda}{2}\right)^2}{\sin^2 \frac{\lambda}{2}}$, if $|\lambda| \leqslant 1$, and $a(\lambda) = 0$, if $|\lambda| > 1$. We will obtain that it is evenly on μ

$$\frac{1}{x} \int_{-1/x}^{1/x} w(\lambda+\mu)\, d\lambda = \frac{1}{\pi} h(x)(1+o(1)). \qquad (4.35)$$

End Section.

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Again let us use lemma 14, but by choosing now $\alpha(\lambda) = (\lambda/2)^2 / \sin^2\frac{\lambda}{2}$ if $0<\lambda\leqslant 1$, and $\alpha(\lambda) = 0$, if $\lambda > 1$, whereupon $\alpha(\lambda) = -\alpha(-\lambda)$, if $\lambda < 0$. We will obtain that it is evenly on μ

$$\frac{1}{x}\int_0^{1/x} w(\lambda+\mu)\,d\lambda - \frac{1}{x}\int_{-1/x} w(\lambda+\mu)\,d\lambda = \\ = o(h(x)) = o\left(\frac{1}{x}\int_{-1/x}^{1/x} w(\lambda+\mu)\,d\lambda\right). \quad (4.36)$$

After designating $W(\lambda)$ original $w(\lambda)$ and after replacing $1/x$ by δ, we let us will be able to rewrite (4.36) in the following equivalent form: it is evenly on μ with $\delta \longrightarrow 0$

$$W(\mu+\delta) + W(\mu-\delta) - 2W(\mu) = o(W(\mu+\delta) - W(\mu-\delta))$$

or

$$W(\mu+\delta) + W(\mu-\delta) - 2W(\mu) = o(W(\mu\pm\delta) - W(\mu)). \quad (4.37)$$

If we use usual designations $\Delta_\delta W(\mu) = W(\mu+\delta) - W(\mu)$, $\Delta_\delta^2 W(\mu) = \Delta_\delta\Delta_\delta W(\mu)$, that, as can easily be seen, (4.37) it is equivalent to the following relationship/ratio: evenly on μ

$$\Delta_\delta^2 W(\mu) = o(\Delta_\delta W(\mu)). \quad (4.38)$$

Finally, using a designation $\omega_W(\delta)$, introduced in the formulation of theorem 5, we see that (4.37) it indicates: $\omega_W(\delta)\to 0$ with $\delta \longrightarrow 0$. Theorem of 5 is proved.

It is below, in §5, we will give some corollaries of this theorem. Now let us pass to the proof of theorem 6.

From theorem 2 it follows that in proof it suffices to be bounded to estimation ρ (τ, w). Therefore let us below consider polynomial P from representation (3.1) equal to 1.

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Let us place $W_1(\lambda) = W(\lambda) + \alpha\lambda$, where a it is selected so, in order to $W_1(\pi) = W_1(-\pi)$. It is obvious,

$$\sup_{\lambda, |x|\leqslant\delta} \frac{|W_1(\lambda+x)+W_1(\lambda-x)-2W_1(\lambda)|}{|W(\lambda+x)-W(\lambda-x)|} = \\ = \sup_{\lambda, |x|\leqslant\delta} \frac{|W(\lambda+x)+W(\lambda-x)-2W(\lambda)|}{|W(\lambda+x)-W(\lambda-x)|} = \omega_W(\delta).$$

Lemma 15. For any trigonometric polynomials $\psi, \varphi \in \mathcal{H}^2$ occurs

the equality $\int_{-\pi}^{\pi} \varphi(\lambda)\psi(\lambda)e^{i\lambda\tau}f(\lambda)\,d\lambda =$

$$= \int_{-\pi}^{\pi} [\varphi'(\lambda)\psi(\lambda) + \varphi(\lambda)\psi'(\lambda) + i\tau\varphi(\lambda)\psi(\lambda)]\,e^{i\tau\lambda}W_1(\lambda)\,d\lambda. \tag{4.39}$$

Proof. For all $\tau > 0$

$$\int_{-\pi}^{\pi} \varphi(\lambda)\psi(\lambda)e^{i\lambda\tau}\,d\lambda = 0.$$

Therefore

$$\int_{-\pi}^{\pi} \varphi(\lambda)\psi(\lambda)e^{i\lambda\tau}f(\lambda)\,d\lambda = \int_{-\pi}^{\pi} \varphi(\lambda)\psi(\lambda)e^{i\lambda\tau}\,dW_1(\lambda).$$

By producing in the last/latter integral integration in parts, we will obtain (4.39).

Subsequently the proof of theorem 6 is realized as follows. Conditions 1), 2) theorem mean that W_1 - sufficiently smooth function and, therefore, it approaches sufficiently well by trigonometric polynomials. Carrying out in (4.39) approach/approximation W_1 by such polynomials, we will obtain estimation for $\rho(\tau)$ (comp. with the proof of theorem 1).

Let $\varphi(\lambda) = \sum_{0}^{n} a_j e^{i\lambda j}$ – be a trigonometric polynomial.

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Under conditions of theorem 6 occur the following inequalities:

(1)

$$\|\varphi'\|_f \leqslant C_1 n \|\varphi\|_f \quad \text{(неравенство Зигмунда)}, \quad (4.40)$$

(2)

$$\left\|\sum_0^m a_j e^{i\lambda j}\right\|_f \leqslant C_2 \|\varphi\|_f, \quad m \leqslant n \quad \text{(неравенство М. Рисса)}, \quad (4.41)$$

Key: (1). (inequality of Zigmunda). (2). (M. Riesz' inequality).

and the inequality of Littulwda - the Peli: for any whole r > 0

$$\left\|\sum_0^r a_j e^{i\lambda j}\right\|_f^2 + \sum_p \left\|\sum_{2^{p-1}r+1}^{2^p r} a_j e^{i\lambda j}\right\|_f^2 \leqslant C_3 \|\varphi\|_f^2 \quad (4.42)$$

(here is set/assumed $a_j = 0$, if $j > n$). Constants C_1, C_2, C_3 depend only on f, but not on ϕ.

On the strength of assumption 0, m, = min f (λ) $\leqslant$ f (λ) $\leqslant$ maxf (λ) = M < ∞ these inequalities are obvious. For example,

$$\|\varphi'\|_f \leqslant \sqrt{M}\|\varphi'\|^{(2)} \leqslant \sqrt{M}\, n \|\varphi\|^{(2)} \leqslant \sqrt{\frac{M}{m}}\, n \|\varphi\|_f.$$

Analogously they are proven inequalities (4.41), (4.42); it is possible to place $C_2 = \sqrt{M/m}$, $C_3 = M/m$.

Lemma 16. Let

$$\varphi(\lambda) = \sum_L^N a_j e^{i\lambda j}, \quad 0 \leqslant L \leqslant N,$$

$$\psi(\lambda) = \sum_{L_1}^{N_1} b_j e^{i\lambda j}, \quad 0 \leqslant L_1 \leqslant N_1, \quad 1 < L_1 + L$$

Then

$$\left|\int_{-\pi}^{\pi}\varphi(\lambda)\psi(\lambda)f(\lambda)\,d\lambda\right|\leqslant \leqslant C_4\left(\frac{M}{m}\right)^{3/2}\frac{N+N_1}{L+L_1-1}\,\omega_W\left(\frac{1}{L+L_1-1}\right). \quad (4.43)$$

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Proof. In accordance with (4.39) it suffices to demonstrate that

$$\begin{gathered}\left|\int_{-\pi}^{\pi}\varphi'(\lambda)\psi(\lambda)W_1(\lambda)\right|\leqslant C_4\left(\frac{M}{m}\right)^{3/2}\frac{N\omega_W\left(\frac{1}{L+L_1-1}\right)}{L+L_1-1},\\ \left|\int_{-\pi}^{\pi}\varphi(\lambda)\psi'(\lambda)W_1(\lambda)\,d\lambda\right|\leqslant \\ \leqslant C_4\left(\frac{M}{m}\right)^{3/2}\frac{N_1}{L+L_1-1}\,\omega_W\left(\frac{1}{L+L_1-1}\right).\end{gathered} \quad (4.44)$$

From the conditions of theorem 6 and determination of function W_1 it follows that

$$|W_1(\lambda+\delta)+W_1(\lambda-\delta)-2W_1(\lambda)|\leqslant 2M\delta\omega_W(\delta).$$

i.e. W_1 (λ) - smooth function [1]), and according to the theorem of N. I. Akhiyezer (see [25], of page 274) value $E_s(W_1)$ the best approximation of function W_1 trigonometric by the polynomials of degree $\leqslant$ s it satisfies the inequality

$$E_s(W_1) \leqslant C_4 \frac{M}{s} \omega_W \left(\frac{1}{s}\right). \tag{4.45}$$

FOOTNOTE [1]. Function W (λ) is called smooth, if its "module/modulus of smoothness" satisfies the condition:

$$\sup_{\lambda,\ |x| \leqslant \delta} | W(\lambda + x) + W(\lambda - x) - 2W(\lambda) | = o(\delta), \quad \delta \to 0.$$

ENDFOOTNOTE.

Let Q - the polynomial of the best approximation for W_1 of degree be not higher than L + L_1 - 1. Then

$$\begin{aligned}
\left| \int_{-\pi}^{\pi} \varphi'(\lambda) \psi(\lambda) W_1(\lambda) \, d\lambda \right| &= \\
&= \left| \int_{-\pi}^{\pi} \varphi'(\lambda) \psi(\lambda) [W_1(\lambda) - Q(\lambda)] \, d\lambda \right| \leqslant \\
&\leqslant \frac{1}{m} E_{L+L_1-1}(W_1) \| \varphi' \|_f \| \psi \|_f \leqslant \\
&\leqslant \frac{C_1 C_4}{m} M \frac{N}{L+L_1-1} \omega_W \left(\frac{1}{L+L_1-1} \right) \leqslant \\
&\leqslant C_1 \left(\frac{M}{m} \right)^{3/2} \frac{N \omega_W \left(\frac{1}{L+L_1-1} \right)}{L+L_1-1}.
\end{aligned}$$

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Analogously is proven the second of inequalities (4.44).

Let us pass to the proof of theorem 6. Let φ, $\psi \in \mathscr{H}^2-$ be trigonometric polynomials from $\|\varphi\|_f = \|\psi\|_f = 1$ the form

$$\varphi(\lambda) = \sum_{\tau+1}^{N} a_j e^{i\lambda j}, \quad \psi(\lambda) = \sum_{0}^{N_1} b_j e^{i\lambda j}.$$

Without loss of generality it is possible to count $N = 2^p\tau,\ N_1 = 2^p\tau,\ p \geqslant 0.$

Let us assume with j = 1, ..., p

$$\varphi_j(\lambda) = \sum_{2^{j-1}\tau+1}^{2^j\tau} a_s e^{i\lambda s},$$

$$\Phi_j(\lambda) = \sum_{\tau+1}^{2^j\tau} a_s e^{i\lambda s},$$

$$\psi_j(\lambda) = \sum_{2^{j-1}\tau+1}^{2^j\tau} a_s e^{i\lambda s}, \quad \psi_0(\lambda) = \sum_{0}^{\tau} a_s e^{i\lambda s},$$

$$\Psi_j(\lambda) = \sum_{0}^{2^j\tau} a_s e^{i\lambda s}.$$

By using inequality (4.43) and by inequalities (4.42), (4.41), let us find

$$\left|\int_{-\pi}^{\pi}\varphi(\lambda)\psi(\lambda)f(\lambda)\,d\lambda\right| \leqslant$$
$$\leqslant \sum_{l=1}^{p}\left(\left|\int_{-\pi}^{\pi}\varphi_l(\lambda)\Psi_l(\lambda)f(\lambda)\,d\lambda\right| + \left|\int_{-\pi}^{\pi}\psi_l(\lambda)\Phi_l(\lambda)f(\lambda)\,d\lambda\right|\right) \leqslant$$
$$\leqslant 8C_4\left(\frac{M}{m}\right)^{3/2}\left[\max_l \|\Psi_l\|_f \sum_{l=1}^{p}\omega_W\left(\frac{1}{2^{l-1}\tau-1}\right)\|\varphi_l\|_f + \right.$$
$$\left. + \max_l \|\Phi_l\|_f \sum_{l=1}^{p}\omega_W\left(\frac{1}{2^{l-1}\tau-1}\right)\|\psi_l\|_f\right] \leqslant$$
$$\leqslant 8C_4\left(\frac{M}{m}\right)^{2}\left(\sum_{l=1}^{\infty}\omega_W^2\left(\frac{1}{2^{l-1}\tau-1}\right)\right)^{1/2}\left(\left(\sum_l \|\varphi_l\|_f^2\right)^{1/2} + \right.$$
$$\left. + \left(\sum_l \|\psi_l\|_f^2\right)^{1/2}\right) \leqslant 6C_4\left(\frac{M}{m}\right)^{5/2}\left(\sum_{l=1}^{\infty}\omega_W^2\left(\frac{1}{2^{l-1}\tau-1}\right)\right)^{1/2}. \tag{4.46}$$

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Taking into account that $C_4 \leqslant 5/2$ [25], page 305), let us arrive at (3.3). Theorem 6 is demonstrated.

§5. Corollaries of the fundamental theorems. Examples.

Let us give several corollaries §§ 2-4, which permit more

graphically to present, which limitations assigns on spectral density the condition of full/total/complete regularity. Let us note first of all that

the spectral density of completely regular process with discrete time does not have discontinuity/interruptions the over/rewater of kind (jumps).

For a proof let us turn to representation (3.1). It suffices to demonstrate that the jumps do not have w (λ). Actually, if w (λ) had a jump at point λ_0, it would be limited in the vicinity of this point. Consequently, in this vicinity function W (λ) would be smooth on the strength of (3.1), i.e., would take place the equality

$$W(\lambda+\delta)+W(\lambda-\delta)-2W(\lambda)=o(\delta).$$

But, as is known ([3], page 182), the derivative of smooth function, possessing Darboux's property, it cannot have first-order discontinuities.

Above has already been noted that in expansion (2.1) or (3.1) everything zero spectral density f (λ) will carry the first polynomial factor, the second factor w (λ) "almost" is positive, so that all zero f (λ), roughly speaking they have the whole even order. In order to give to the aforesaid precise sense, let us note first

that

functions w from expansions (2.1), (3.1) are summarized with any degree of p, $-\infty < p < \infty$.

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Let w - from expansion (2.1). It is obvious, with all $\varepsilon > 0 \| \exp\{\pm r \pm u_\varepsilon\} \|^{(\infty)} \leqslant \| \exp\{|r_\varepsilon| + |u_\varepsilon|\} \|^{(\infty)} < \infty$. Above we already noted (page 212) that from inequality $\| v_\varepsilon \|^{(\infty)} < \varepsilon$ it follows that for all $k < \frac{\pi}{2\varepsilon} \exp\{k | \tilde{v}_\varepsilon |\} \in \mathscr{L}^1$ and, which means, $e^{\pm \tilde{v}_\varepsilon} \in \mathscr{L}^p$, if only $\varepsilon < \frac{\pi}{2p}$. Further, if now w - from expansion (3.1), then (as it follows from the proof of theorem 5) $w^{-1} \in \mathscr{L}^1$. Equate/comparing expansions (2.1) and (3.1), we find hence that in both expansions function w is one and the same.

Let us name order, the upper order, lower order zero λ_0 of function f (λ) respectively of the number

$$k(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{\ln f(\lambda)}{\ln |\lambda - \lambda_0|}, \quad \bar{k}(\lambda_0) = \overline{\lim_{\lambda \to \lambda_0}} \frac{\ln f(\lambda)}{\ln |\lambda - \lambda_0|},$$

$$\underline{k}(\lambda_0) = \underline{\lim_{\lambda \to \lambda_0}} \frac{\ln f(\lambda)}{\ln |\lambda - \lambda_0|}.$$

From recently the demonstrated proposition relative to summability w it follows that

The lower order of any zero w is equal to zero, so that the present order zero f (λ) can be only whole even.

Finally, it is obvious that

spectral density f (λ) completely regular process $\xi(t)$ is summarized with any positive degree.

let us now move on to the construction of examples of completely regular processes, more precise, spectral densities f (λ) of such processes. it goes without saying that the large number of examples can be constructed with the help of theorem 1. Significant examples will be obtained, only if are not observed the conditions of theorem 1, i.e., if f (λ) can be converted into zero not by polynomial form or have discontinuity/interruptions (of course, the second kind); similar examples can be constructed with the help of theorems 3 and 6.

Preliminarily let us establish the following result, after designating $\widetilde{\ln f}$ adjoint function and $\ln f \in \mathscr{L}^1(-\pi, \pi)$.

Theorem 7. If at least one of the functions lnf, $\widetilde{\ln f}$ is continuous, stationary process $\xi(t)$ with spectral density f (λ) completely regular, whereupon

$$\rho(\tau) = O\left(\min \omega\left(\frac{1}{\tau}; \ln f\right), \omega\left(\frac{1}{\tau}; \widetilde{\ln f}\right)\right), \qquad (5.1)$$

where ω (δ; h) it designates the module/modulus of the continuity of function h (λ).

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Proof. In the case of the continuity of function lnf (λ) it suffices to exile to theorem 1, since according to Jackson's theorem about the best approximations $E_{\tau-1}(\ln f)=O\left(\omega\left(\frac{1}{\tau};\ln f\right)\right)$. The full/total/complete regularity of process $\xi(t)$ with the continuous function $\widetilde{\ln f}$ follows from theorem 3. Actually, whatever number $\xi(t)$ it is possible, relying on $\varepsilon>0$, the Weyerstrass theorem, to find trigonometric polynomial Q_ε by such that

$$\widetilde{\ln f}=Q_\varepsilon+v_\varepsilon,\quad \|v_\varepsilon\|^{(\infty)}\leqslant\varepsilon.$$

Obviously, function $r_\varepsilon=\tilde{Q}_\varepsilon$ also is a trigonometric polynomial, and therefore for all $\varepsilon>0$ f (λ) it allow/assumes the representation

$$f(\lambda)=e^{\ln f(\lambda)}=e^{r_\varepsilon+\theta_\varepsilon},\quad \|v_\varepsilon\|^{(\infty)}\leqslant\varepsilon.$$

Now already it is possible to exile to theorem 3.

However in order to obtain estimation (5.1), it is required to lead the direct/straight proof, however, very simple.

Let this time Q (λ) there is a trigonometric polynomial of the best approximation of degree $\leqslant \tau-1$ for continuous function $e^{i\widetilde{\ln f}}$.

Then with all $\theta \in \mathscr{H}^1$

$$\int_{-\pi}^{\pi} e^{i\lambda\tau}\theta(\lambda)\,Q(\lambda)\,d\lambda = 0.$$

So that on the strength of (1.7)

$$\rho(\tau) = \sup_{\theta \in H^1,\ \|\theta\|^1 = 1} \left| \int_{-\pi}^{\pi} e^{i\lambda\tau}\theta(\lambda)\,\frac{\overline{g(e^{i\lambda})}}{g(e^{i\lambda})}\,d\lambda \right| =$$

$$= \sup_{\theta} \left| \int_{-\pi}^{\pi} e^{i\lambda\tau}\theta(\lambda)\,[e^{i\,\widetilde{\ln f}} - Q]\,d\lambda \right| \leqslant$$

$$\leqslant E_{\tau-1}(e^{i\,\widetilde{\ln f}}) = O\left(\omega\left(\frac{1}{\tau};\ e^{i\,\widetilde{\ln f}}\right)\right) = O\left(\omega\left(\frac{1}{\tau};\ \widetilde{\ln f}\right)\right).$$

Theorem is demonstrated.

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Let us give several examples, based on this theorem.

Example. Let $\xi(t)$ there is a stationary process with the spectral density

$$f(\lambda) = \exp\left\{\sum_{1}^{\infty} \frac{\cos k\lambda}{k(\ln k + 1)}\right\}.$$

In this case the function $\widetilde{\ln f}(\lambda) = \sum_{1}^{\infty} \frac{\sin k\lambda}{k(\ln k + 1)}$ is continuous, and according to theorem 7 process $\xi(t)$ completely regular. At the same time, using the properties of Fourier series with monotonic to coefficients [1], it is possible to show that with $\lambda \to 0$

FOOTNOTE [1]. See [13], page 303. ENDFOOTNOTE.

$$\ln f(\lambda) \sim \int_1^{1/\lambda} \frac{dt}{t(\ln t + 1)} \sim \ln \ln \frac{1}{\lambda}.$$

Consequently, with $\lambda \longrightarrow 0$ the function f (λ) grows as $\ln 1/\lambda$. Analogously process $\xi(t)$ with spectral density 1/f (λ) also is completely regular, but now its spectral density in zero is converted into zero as $1/\ln 1/\lambda$.

It is possible to count that the module/modulus of continuity ω (δ) function lnf in this example does not exceed $\widetilde{\ln f}$ so that

$$O\left(\frac{1}{\ln 1/\delta}\right), \text{ so that}$$

$$\rho(\tau) = O\left(\frac{1}{\ln \tau}\right).$$

Example. Let us give this time an example of the completely regular process, spectral density of which is limited on top and from below, but it has discontinuity/interruptions (it goes without saying that the second kind). For this in accordance with theorem 7 it suffices to construct the limited discontinuous function, conjugated/combined with which is continuous.

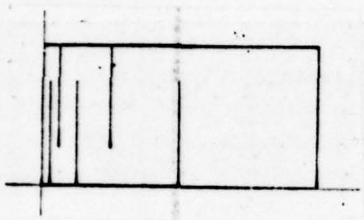

Fig. 2.

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For this purpose let us consider the given in Fig. 2 rectangular region with cut/sections, "J. Littlvud's hippopotamus" [1].

FOOTNOTE [1]. Sm. J. Littlvud mathematical mixture, Fizmatgiz, 1962. The regions of this type are well known in the theory of principal points, see [9]. ENDFOOTNOTE.

The number of cut/sections - teeth - is infinite, the length of each cut/section 3/4. Let G (z) - the function, which conformally reflects

is unit circle to this region. If $G(e^{i\lambda})$ = u (λ) + IV (λ), then, obviously, function v (λ) is disruptive, $0 \leqslant v \leqslant 1$, and function u (λ) = $\tilde{v}$ (λ) is continuous. We can select as the unknown spectral density function $f(\lambda)=e^{v(\lambda)}$. It would be interesting to indicate sufficiently good estimation for ρ (τ).

§6. Rapid mixing.

Until now, we were occupied by the investigation of the question concerning convergence ρ (τ) to zero, being barely interested in rate of convergence. Here we will be occupied this problem, after placing by the target/purpose of explaining, when ρ (τ) decreases with τ → ∞ fast enough, let us say, not slower $\tau^{-\varepsilon}$, $\varepsilon>0$ (let us note that from the viewpoint of the applicability of central limit theorem to the process, which satisfies the condition of powerful mixing, this is the most interesting case) [2].

FOOTNOTE [2]. See [14], chapter 18 or [22], chapter 4. ENDFOOTNOTE.

It proves to be, here it is possible to obtain the sufficiently

complete and final results.

Theorem 8. In order that

$$\rho(\tau)=O(\tau^{-r-\beta}),\ \text{где}\ 0<\beta<1,$$

it is necessary and sufficiently in order that spectral density f (λ) would allow/assume representation of the form

$$f(\lambda)=|P(e^{i\lambda})|^2 w(\lambda), \tag{6.1}$$

where P (z) - polynomial with zero on $|z|=1$, and function w (λ) is strictly positive, w (λ) $\geqslant$ m > 0, r once differentiated and it r-I derivative satisfies the condition of Gelder of order β.

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Proof. On the basis of theorem 3 or 5 spectral density f (λ) can be presented in the form (3.1), where w (λ) satisfies condition (3.2). In this case according to theorem 2 and to lemma 11

$$\begin{gathered}\rho(n;\ f)\leqslant\rho(n-k;\ w),\\ \rho(n;\ w)\leqslant\rho(n;\ f),\end{gathered} \tag{6.2}$$

where k is a degree of polynomial P. Therefore it suffices to be bounded to those by the case, when f ≡ w, i.e., when in representation (3.1) polynomial P is absent and already spectral density itself f (λ) satisfies condition (3.2). Below, without

specifying this especially, we everywhere set/assume P ≡ 1.

The sufficiency of the conditions of theorem escape/ensues immediately from theorem 1. Actually, according to this theorem

$$\rho(\tau;\ f)=\rho(\tau)\leqslant\frac{1}{m}E_{\tau-1}(f). \qquad (6.3)$$

On the other hand, as it is claimed in Jackson's theorem about the order of the best approximations [1],

$$E_{\tau-1}(f)\leqslant\frac{C}{\tau^{r+\beta}}. \qquad (6.4)$$

FOOTNOTE [1]. See [25], page 275. ENDFOOTNOTE.

Need. That that spectral density f (λ) of process with the rapidly decreasing coefficient of regularity must be sufficiently smooth, has already been explained in general terms on page 207. It was shown, that the smallness of the coefficient of Fourier's regularity function f (λ). The used there simple idea is the basis of our present examinations. Only instead of the Fourier coefficients to advantageously consider immediately the segments of Fourier series f (λ). The obtained results in turn, will be used to evaluate the value of the best approximations of function f (λ) by trigonometric polynomials. Finally, the converse theorems of Bernstein from the theory of the best approximations allow, on the basis of the order of

the best approximations of function f (λ), to draw the determined conclusions about its smoothness. Here is the exemplary/approximate diagram of proof.

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Let r_j designate the j Fourier coefficient function f (λ). On the strength of (1.5) occurs the inequality

$$\left|\sum_{2^k}^{2^{k+1}} r_j e^{ij\mu}\right| = \left|\int_{-\pi}^{\pi}\left(\sum_{2^k}^{2^{k+1}} e^{ij(\mu-\lambda)}\right) f(\lambda)\,d\lambda\right| \leqslant$$
$$\leqslant \rho(2^k)\int_{-\pi}^{\pi}\left|\sum_{0}^{2^k} e^{ij(\mu-\lambda)}\right| f(\lambda)\,d\lambda, \quad k \geqslant 0.$$

If we assume it is additional that $\max_\lambda |f(\lambda)| \leqslant M < \infty$, that let us have

$$\left|\sum_{2^k}^{2^{k+1}} r_j e^{ij\mu}\right| \leqslant \rho(2^k) M \int_{-\pi}^{\pi}\left|\frac{\sin 2^{k-1}}{\sin\frac{\lambda}{2}}\right| d\lambda \leqslant C_1\rho(2^k)k.$$

Analogous inequalities, obviously, occur also for negative k.

If now $2^k \leqslant \tau < 2^{k+1}$, that the value of the best approximation f (λ) by the trigonometric polynomials of degree τ knowingly does not exceed

$$\max_\mu\left|\sum_{|j|\geqslant 2^k} r_j e^{i\mu j}\right| \leqslant \sum_{p=k}^{\infty}\left|\sum_{2^p\leqslant|j|<2^{p+1}} r_j e^{i\mu j}\right| \leqslant$$
$$\leqslant C_2 \sum_{p\geqslant \ln_2 \tau+1} k\rho(2^k).$$

From the obtained inequality almost immediately follows theorem 8; true, we in state to guarantee for $f^{(r)}(\lambda)$ Gelder's condition with index $\beta-\varepsilon$, where ε as conveniently little (but not with β).

In order to get rid of this ε, it is necessary to make sufficiently cumbersome calculations. During this instead of the sums Fourier's lifetime will be considered Fejer's truncated sums for f (λ) (examine them completely logically, since they approach f (λ), actually, so accurately as polynomials of the best approximation).

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Lemma 17. With all θ, $0 < \theta < 1$, occur of the inequality

$$E_n(f) \leqslant \frac{32 \max\limits_{\lambda} f(\lambda)}{\theta^2} \sum_{k=0}^{\infty} \rho(2^k n(1-\theta); f). \qquad (6.5)$$

Proof. Sufficient to consider only that case, when a series in right side (6.5) descends; if this series diverges, inequality (6.5) is trivial.

Let us demonstrate first that if a series $\sum\rho(2^k)$ descends, then function f (λ) is limited. For this again let us turn to function γ (N; μ) from §§3, 4 let us demonstrate that γ (N; μ) $\leqslant NM$, where constant $M < \infty$ does not depend on N and μ.

In the proof of lemma 8 was obtained equality (3.14). If we assume in this equality k = 3, it is not difficult to count, that

$$\gamma(3N;\mu)\leqslant 3\gamma(N;\mu)+9\sqrt{\gamma(N;\mu)\gamma(r;\mu)}+9\gamma(N;\mu)\rho(r).$$

Whatever number $\varepsilon>0$, with large N evenly on μ (see §3)

$$N^{1-\varepsilon}\leqslant\gamma(N;\mu)\leqslant N^{1+\varepsilon}.$$

After selecting here $\varepsilon=\frac{\ln 3-\ln 2}{2\ln 6}$, we will obtain that with large N it is evenly on μ

$$\frac{\gamma(3N;\mu)}{3\gamma(N;\mu)}\leqslant 1+3\rho(r)+3\frac{r^{\frac{1+\varepsilon}{2}}}{N^{\frac{1-\varepsilon}{2}}}.$$

set/assuming finally $N=3^s$, s = 1, 2, ...; r = r (N) = 2^s let us have for large s evenly on μ

$$\frac{\gamma(3^{s+1};\mu)}{3\gamma(3^s;\mu)}\leqslant 1+3\rho(2^s)+3\left(\frac{2}{3}\right)^{s/4}.$$

Consequently,

$$\lim_{s\to\infty}\frac{\gamma(3^s;\mu)}{3^s}\leqslant C\prod_1^\infty\left(1+3\rho(2^s)+3\left(\frac{2}{3}\right)^{s/4}\right)=M<\infty,$$

since

$$\sum_s \left[\rho(2^s) + \left(\frac{2}{3}\right)^{s/4}\right] < \infty.$$

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Thus, we demonstrated the existence of constant M to such that

$$3^{-s}\gamma(3^s;\,\mu) \leqslant M. \qquad (6.6)$$

According to Lebesgue's theorem for almost all $\mu \in [-\pi,\,\pi]$

$$\lim_{N\to\infty} \frac{1}{N}\gamma(N;\,\mu) = \lim_N \frac{1}{N} \int_{-\pi}^{\pi} \frac{\sin^2 N\,\frac{\lambda-\mu}{2}}{\sin^2 \frac{\lambda-\mu}{2}} f(\lambda)\,d\lambda = 2\pi f(\mu),$$

and on the strength of (6.6) for almost all μ f $(\mu) \leqslant M/2\pi$. Consequently, it is possible to count that sup f $(\mu) \leqslant$ M.

Subsequently let us place $M = \sup_\mu f(\mu)$. Let r_j there is the j Fourier coefficient function f (λ), by $S_n(\lambda) = \sum_{-n}^{n} r_j e^{ij\lambda}$ let us designate n-s the particular summation of series of Fourier function f (λ). Let us take finally, any number θ, $0 < \theta < 1$, and let us introduce of examination Fejer's truncated sums for function f (λ):

$$\sigma_n(f;\,[n\theta];\,\lambda) = \frac{1}{[n\theta]+1} \sum_{\nu=0}^{[n\theta]} S_{n-\nu}(\lambda).$$

It is obvious,

$$E_n(f) \leqslant \max_\lambda |g(\lambda) - \sigma_n(f;\,[n\theta];\,\lambda)|.$$

We will show that

$$\max_{\lambda} | g(\lambda) - \sigma_n(f;\ [n\theta];\ \lambda) | \leqslant C(\theta) \sum_{k=0}^{\infty} \rho([2^k n(1-\theta)]),$$

where C (θ) - is constant, that depends on θ. Let us consider for this purpose of the equality

$$\Delta_k(\lambda) = \sigma_{2^{k+1}n}(f;\ [2^{k+1}n\theta];\ \lambda) - \sigma_{2^k n}(f;\ [2^k n\theta];\ \lambda).$$

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It is easy to see that these differences can be registered in the form

$$\Delta_k(\lambda) = \Delta_k^+(\lambda) + \Delta_k^-(\lambda),$$

where the sum Δ_k^+ and Δ_k^- they take the following form:

$$\Delta_k^+(\lambda) = \sum_{[2^k n\theta]+1}^{2^{k+1}n} a_l r_l e^{il\lambda},$$
$$\Delta_k^-(\lambda) = \sum_{-2^{k+1}n}^{-[2^k n\theta]-1} a_l r_l e^{il\lambda},$$

a coefficients a_l do not depend on function f (λ) (their precise form for us is unessential).

Simple calculations make it possible to rewrite Δ_k^+ and Δ_k^- in the form

$$\Delta_k^+(\lambda)=\int_{-\pi}^{\pi} r_k^+(\lambda-\mu)\,f(\mu)\,d\mu,$$
$$\Delta_k^-(\lambda)=\int_{-\pi}^{\pi} r_k^-(\lambda-\mu)\,f(\mu)\,d\mu,$$

where

$$r_k^+(\lambda)=\frac{1}{[2^{k+1}n\theta]+1}\sum_{\nu=0}^{[2^{k+1}n\theta]}\frac{1}{2\pi}\sum_{l=1}^{2^{k+1}n-\nu}e^{il\lambda}-\frac{1}{[2^kn\theta]+1}\sum_{\nu=0}^{[2^kn\theta]}\frac{1}{2\pi}\sum_{l=1}^{2^kn-\nu}e^{il\lambda},$$
$$r_k^-(\lambda)=\overline{r_k^+(\lambda)}.$$

The trigonometric polynomial $r_k^+(\lambda-\mu)$, considered as function μ, contains only negative degrees $e^{i\mu}$, beginning from $\exp\{-(2^kn-[2^kn\theta]+1)\mu\}$. Therefore (see equality (1.5))

$$|\Delta_k^+(\lambda)|\leqslant\rho(2^kn-[2^kn\theta]+1)\int_{-\pi}^{\pi}|r_k^+(\lambda-\mu)|f(\mu)\,d\mu\leqslant$$
$$\leqslant M\rho(2^kn-[2^kn\theta]+1)\left[\int_{-\pi}^{\pi}|\operatorname{Re}r_k^+(\lambda-\mu)|\,d\mu+\int_{-\pi}^{\pi}|\operatorname{Im}r_k^+(\lambda-\mu)|\,d\mu\right].\qquad(6.7)$$

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It is analogous

$$|\Delta_k^-(\lambda)| \leqslant M\rho(2^k n - [2^k n\theta] + 1)\left[\int_{-\pi}^{\pi} |\operatorname{Re} r_k^-(\lambda-\mu)|\,d\mu + \right.$$
$$\left. + \int_{-\pi}^{\pi} |\operatorname{Im} r_k^-(\lambda-\mu)|\,d\mu\right]. \quad (6.8)$$

Let us show that the integrals in the right side of these inequalities are limited.

1) **Limitedness** $\int_{-\pi}^{\pi} |\operatorname{Re} r_k^+(\lambda-\mu)|\,d\mu.$ Let us for a brevity write N instead of $2^k n$, **p instead of θN. Then after** simple calculations we obtain

$$\operatorname{Re} r_k^+(\lambda-\mu) =$$
$$= \frac{1}{2\pi \sin\frac{\lambda-\mu}{2}}\left[\frac{1}{[2p]+1}\sin\frac{4N+1-[2p]}{2}(\lambda-\mu)\sin\frac{[2p]+1}{2}\times\right.$$
$$\left.\times(\lambda-\mu) - \frac{1}{[p]+1}\sin\frac{2N+1-[p]}{2}(\lambda-\mu)\sin\frac{[p]+1}{2}(\lambda-\mu)\right].$$

With all whole s Fejer's integral

$$\frac{1}{2\pi s}\int_{-\pi}^{\pi}\frac{\sin^2\frac{s}{2}(\lambda-\mu)}{\sin^2\frac{\lambda-\mu}{2}}\,d\mu = 1,$$

therefore

$$\int_{-\pi}^{\pi} |\operatorname{Re} r_k^+(\lambda-\mu)|\,d\mu \leqslant$$
$$\leqslant \frac{(4N+1-[2p])^{1/2}([2p]+1)^{1/2}}{[2p]+1} + \frac{(2N+1-p)^{1/2}(p+1)^{1/2}}{([p]-1)} \leqslant \frac{6}{\theta}.$$

2) Limitedness $\int_{-\pi}^{\pi} |\operatorname{Im} r_k^+ (\lambda - \mu)| \, d\mu.$

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This time we have, retaining as before designation $2^k n = N$,

$$\operatorname{Im} r_k^+ (\lambda - \mu) = \frac{1}{2\pi([2N\theta] + 1)} \sum_{\nu=0}^{[2N\theta]} \sum_{j=1}^{2N-\nu} \sin j (\lambda - \mu) - \\ - \frac{1}{2\pi([N\theta] + 1)} \sum_{\nu=0}^{[N\theta]} \sum_{j=1}^{N-\nu} \sin j (\lambda - \mu).$$

But

$$\sum_{j=1}^{T} \sin j\lambda = \frac{\cos \frac{\lambda}{2} - \cos \frac{2T+1}{2} \lambda}{2 \sin \frac{\lambda}{2}},$$

$$\sum_{T=a}^{b} \cos \left(T + \frac{1}{2}\right) \lambda = \frac{\sin \frac{b-a+1}{2} \lambda \cos \frac{b+a+1}{2} \lambda}{\sin \frac{\lambda}{2}}.$$

So that

$$
\begin{aligned}
&|\operatorname{Im} r_k^+(\lambda-\mu)| = \\
&= \left| -\frac{1}{4\pi([2N\theta]+1)} \frac{\sin\frac{[2N\theta]}{2}(\lambda-\mu)}{\sin^2\frac{\lambda-\mu}{2}} \cos\frac{4N-[2N\theta]+1}{2}(\lambda-\mu) + \right. \\
&\left. + \frac{1}{4\pi([N\theta]+1)} \frac{\sin\frac{[N\theta]+1}{2}(\lambda-\mu)}{\sin^2\frac{\lambda-\mu}{2}} \cos\frac{2N-[N\theta]+1}{2}(\lambda-\mu) \right| \leqslant \\
&\leqslant \frac{1}{2\pi([2N\theta]+1} \frac{\left|\sin\frac{[N\theta]+1}{2}(\lambda-\mu)\right|}{\sin^2\frac{\lambda-\mu}{2}} \sin^2\frac{4N-[2N\theta]+1}{4}(\lambda-\mu) + \\
&+ \frac{1}{2\pi([N\theta]+1)} \frac{\left|\sin\frac{[N\theta]+1}{2}(\lambda-\mu)\right|}{\sin^2\frac{\lambda-\mu}{2}} \sin^2\frac{2N-[N\theta]+1}{4}(\lambda-\mu) + \\
&+ \left| \frac{\sin\frac{[2N\theta]+1}{2}(\lambda-\mu)}{4\pi([2N\theta]+1)\sin^2\frac{\lambda-\mu}{2}} - \frac{\sin\frac{[N\theta]+1}{2}(\lambda-\mu)}{4\pi([N\theta]+1)\sin^2\frac{\lambda-\mu}{2}} \right|. \quad (6.9)
\end{aligned}
$$

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Integral from of the first two terms by right side (6.9) will not exceed 5/40. The last/latter term in (6.9) let us designate for a brevity by b (λ). For $|\lambda| \leqslant 1/2$ $|\sin\lambda - \lambda| \leqslant |\lambda|^3$, and therefore

$$
\begin{aligned}
&\int\limits_{\left\{|\lambda-\mu| \leqslant \frac{1}{[2N\theta]+1}\right\}} b(\lambda)\, d\lambda \leqslant \\
&\leqslant \frac{1}{2\pi} \int\limits_{\left\{|\lambda-\mu| \leqslant \frac{1}{[2N\theta]+1}\right\}} \left[\frac{\left(\frac{1}{2}(2N\theta+1)|\lambda-\mu|\right)^3}{[2N\theta+1]} + \right. \\
&\left. + \frac{\left(\frac{1}{2}(N\theta+1)|\lambda-\mu|\right)^3}{[N\theta]+1} \right] \frac{d\lambda}{\sin^2\frac{\lambda-\mu}{2}} \leqslant 1 < \frac{1}{\theta}.
\end{aligned}
$$

In region $\pi \geqslant |\lambda - \mu| \geqslant \frac{1}{[2N\theta]+1}$ let us have

$$b(\lambda) \leqslant \frac{\left|\sin \frac{[2N\theta]+1}{2}(\lambda-\mu)\right|}{4\pi \sin^2 \frac{\lambda-\mu}{2}} \left| \frac{1}{[2N\theta]+1} - \frac{1}{2([N\theta]+1)} \right| +$$

$$+ \frac{1}{8\pi \left(\sin^2 \frac{\lambda-\mu}{2}\right)([N\theta]+1)} \times$$

$$\times \left| \sin \frac{[2N\theta]+1}{2}(\lambda-\mu) - \sin([N\theta]+1)(\lambda-\mu) \right| +$$

$$+ \frac{1}{4\pi \left(\sin^2 \frac{\lambda-\mu}{2}\right)([N\theta]+1)} \times$$

$$\times \left| \sin \frac{[N\theta]+1}{2}(\lambda-\mu) \cos \frac{[N\theta]+1}{2}(\lambda-\mu) - \right.$$

$$\left. - \sin \frac{[N\theta]+1}{2}(\lambda-\mu) \right| \leqslant \frac{5}{2\pi\theta^2} + \frac{3}{2\pi\theta^2} +$$

$$+ \frac{\sin^2 \frac{[N\theta]+1}{4}(\lambda-\mu)}{2\pi([N\theta]+1)\sin^2 \frac{\lambda-\mu}{2}},$$

and therefore

$$\int\limits_{\left\{|\lambda-\mu| \geqslant \frac{1}{[2N\theta]+1},\ |\lambda| \leqslant \pi\right\}} b(\lambda)\, d\lambda \leqslant \frac{5}{\theta^2} + \frac{4}{\theta}.$$

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Finally we have

$$\int\limits_{-\pi}^{\pi} |\operatorname{Im} r^{+}(\lambda-\mu)|\, d\mu \leqslant \frac{10}{\theta^2}.$$

Further, by gathering the obtained estimations, let us find that

$$\int_{-\pi}^{\pi} |r^{+}(\lambda-\mu)|\,d\mu \leqslant \frac{16}{\theta^2},$$
$$\int_{-\pi}^{\pi} |r^{-}(\lambda-\mu)|\,d\mu \leqslant \frac{16}{\theta^2}. \qquad (6.10)$$

Being returned to the estimation of $\Delta_k(\lambda)$, is concluded from (6.7) and (6.10), that

$$|\Delta_k(\lambda)| \leqslant C(\theta)\rho(2^k n - [2^k n\theta] + 1) \leqslant$$
$$\leqslant C(\theta)\rho(2^k [n(1-\theta)]), \qquad (6.11)$$

where

$$C(\theta) \leqslant \frac{32}{\theta^2} M.$$

Since a series $\sum \rho(2^k)$ **descends, the last/latter inequality means that a series** $\sum \Delta_k(\lambda)$, **a that** means and **sequence** $\{\sigma_{2^k n}(f;\ [2^k n\theta];\ \lambda)\}$ **evenly they descend. but the sequence of Fejer's truncated sums** $\sigma_{2^k n}(f;\ [2^k n\theta)];\ \lambda)$ **for almost all λ converge to f (λ)** [1]**, and consequently, f (λ) is a uniform limit of polynomials** $\sigma_{2^k n}(f;\ 2^k n\theta];\ \lambda)$.

FOOTNOTE [1]. See, for example, [25], page 533. ENDFOOTNOTE.

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By summarizing finally inequalities (6.11), we let us find that

$$E_n(f) \leqslant \max_{\lambda} |f(\lambda) - \sigma_{2^k n}(f;\ [2^k n\theta];\ \lambda)| \leqslant$$
$$\leqslant \sum |\Delta_k(\lambda)| \leqslant \frac{32M}{\theta^2} \sum_{k=0}^{\infty} \rho(2^k [n(1-\theta)]).$$

Lemma 17 is demonstrated.

Theorem 8 is an almost immediate consequence recently of the demonstrated lemma. Actually, in accordance with inequality (6.5) and by the assumption of the theorem about the rate of decrease ρ (τ), it is possible to claim that

$$E_n(f) \leqslant \frac{\text{const}}{n^{r+\beta}\theta^2(1-\theta)^{r+\beta}} \sum_{k=0}^{\infty} 2^{-k(r+\beta)} = O(n^{-r-\beta}). \quad (6.12)$$

According to the converse theorems of the theory of approach/approximation [1]), from inequalities (6.12) it follows that function f (λ) r once is differentiated, a $f^{(r)}$ (λ) satisfies the

condition of Gelder of order β.

To us it remained to demonstrate the positiveness of function f (λ) (recall that $P \equiv 1$!). Let us assume that f (λ) is converted into zero at point λ_0. Function f (λ) satisfies the condition of Gelder of order $\geqslant\beta$ ($|f(\lambda_1)-f(\lambda_2)|\leqslant C|\lambda_1-\lambda_2|^\beta$, so that $|f(\lambda)|\leqslant C|\lambda-\lambda_0|^\beta$). Thus, lower order of zero λ_0 is

$$\underline{k}(\lambda_0)=\varliminf_{\lambda\to\lambda_0}\frac{|\ln f(\lambda)|}{|\ln|\lambda-\lambda_0||}\geqslant\beta>0,$$

that it contradicts the assertion on page 250, according to which $\underline{k}$ (λ_0) = 0. Theorem 8 is demonstrated completely.

Let us explain now, when the coefficient of regularity decreases even more rapidly, namely it is exponential rapidly.

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Theorem 9. In order that

$$\varlimsup_{\tau\to\infty}\sqrt[\tau]{\rho(\tau)}\leqslant e^{-\delta}, \text{ where } 0<\delta<\infty,$$

it is necessary and sufficiently in order that spectral density f (λ) would allow/assume analytical continuation into band$-\delta<\operatorname{Im}\zeta<\delta$ the values of complex variable $\zeta=\lambda+i\mu$.

Proof. Let function f (λ) be analytical in the band indicated. On segment $-\pi \leqslant \lambda \leqslant \pi$ it has perhaps only finite number of zeros. Let $|P(z)|^2$ be a polynomial of degree k, everything zero which they coincide with real zero f (λ). Then $f(\lambda)=|P(e^{i\lambda})|^2 w(\lambda)$, where function w ($\lambda$) strictly is positive on real axis allow/assumes analytical continuation into the same band, as f (λ). It is not difficult to comprehend that this analytical continuation, let us say $w(\zeta)$, limitedly in any band of form $-\delta'\leqslant \operatorname{Im}\zeta\leqslant\delta',\ \delta'<\delta$.

According to S. N. Bershteyn's theorem [1] $\overline{\lim\limits_{\tau\to\infty}}\sqrt[\tau]{E_\tau(w)}\leqslant e^{-\delta}$, a then in accordance with inequality (6.3) and

$$\overline{\lim}\sqrt[\tau]{\rho(\tau;\ w)}\leqslant e^{-\delta}.$$

1) Footnote: See [1], page 235. End Footnote

That means and

$$\overline{\lim_{\tau}}\sqrt[\tau]{\rho(\tau;\ f)}\leqslant\overline{\lim_{\tau}}\sqrt[\tau]{\rho(\tau-k;\ w)}\leqslant e^{-\delta}.$$

Conversely, let $\overline{\lim\limits_{\tau}}\sqrt[\tau]{\rho(\tau;\ f)}\leqslant e^{-\delta}$. Then $\rho(\tau;\ f)\leqslant e^{-\delta\tau}(1+\varepsilon(\tau))^{\tau}$, where $\lim\limits_{\tau} \varepsilon(\tau)=0$. In accordance with lemma 17

$$\overline{\lim_{\tau}}\,[E_\tau(f)]^{1/\tau}\leqslant\lim_{\tau}\Big[\frac{CM}{\theta^2}\sum_{k=0}^{\infty}\exp\{-[\tau(1-\theta)]\delta 2^{-k}\}\times$$
$$\times(1+\varepsilon([\tau(1-\theta)]2^k))^{[\tau(1-\theta)]2^k}\Big]^{1/\tau}=e^{-(1-\theta)\delta}.$$

Number θ here is arbitrary, provided $0<\theta<1$, and therefore $\overline{\lim}\,[E_\tau(f)]^{1/\tau}\leqslant e^{-\delta}.$

The converse theorem of S. N. Bernstein claims that function f (λ) analytically continuable into band $-\delta<\operatorname{Im}\zeta<\delta$ (is limited in any band of form $-\delta'\leqslant\operatorname{Im}\zeta\leqslant\delta,\ \delta'<\delta$). Theorem is demonstrated.

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Analogously it is proven.

Theorem 10. In order that ρ (τ) = $O(e^{-\tau\delta})$ – with everyone δ > 0, it is necessary and sufficient in order that the analytical continuation f (λ) would be the whole complex variable function $\zeta=\lambda+i\mu$.

Let us note also the following result.

Theorem 11. In order that ρ (τ) = 0 with all $\tau \geqslant k+1$, it is necessary and sufficient in order that f (λ) = $|P(e^{i\lambda})|^2$, where P (z) - the polynomial of degree is not higher than k.

Sufficiency is obvious. Need escape/ensues from the inequality

$$\left|\int_{-\pi}^{\pi} e^{i\lambda s} f(\lambda)\, d\lambda\right| \leqslant \rho(s) \int_{-\pi}^{\pi} f(\lambda)\, d\lambda = 0,$$

if $|s|\geqslant k+1$.

Observation 1. To theorem 8 it is possible to give the more final form, after solving index β to accept and value 1, if we instead of the theorems of Jackson and Bernstein we refer to the more precise result, which evaluates the value of the best approximation through the module/modulus of smoothness [1].

FOOTNOTE [1]. See [25], page 275. ENDFOOTNOTE.

Refined theorem 8 appears as follows.

Theorem 8'. For that in order to $\rho(\tau) = O(\tau^{-r-\beta})$ with certain β, $0 < \beta \leqslant 1$, it is necessary and sufficient in order that f (λ) would be represented in the form

$$f(\lambda) = |P(e^{i\lambda})|^2 w(\lambda),$$

where w (λ) is strictly positive, has r derivative and evenly in terms of μ

$$\max_{h \leqslant \delta} |w^{(r)}(\lambda + h) + w^{(r)}(\lambda - h) - 2w^{(r)}(\lambda)| = O(\delta^{\beta}).$$

Observation 2. From lemma 7 escape/ensues the following useful proposition:

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if $\sum_{k=0}^{\infty} \rho(2^k) < \infty$, that spectral density f (λ) is continuous.

end section.

Page 266. Chapter VI.

FULL/TOTAL/COMPLETE REGULARITY. Processes with continuous time.

§1. Introduction.

In this chapter will be studied the spectral characteristics of completely regular stationary processes $\xi(t)$ with the continuous time t, $-\infty < t < \infty$. In accordance with that which was presented in §1 chapter IV, the obtained in this direction results will characterize also the spectrum of the Gaussian stationary processes, which satisfy the condition of powerful mixing. Let us note previously that the results, which concern the behavior of spectral densities f (λ) of completely regular processes with continuous time in any finite interval of the variation λ, completely analogous to the results of chapter V, demonstrated for processes with discrete time. The specific difficulties appear, only when matter it concerns behavior f (λ) with $\lambda \longrightarrow \infty$; unfortunately, precisely in this point/item our

analysis it makes less exhausting. Certain concept about the appearing here phenomena give theorems 5 and 6 (see below §§5,6).

Any completely regular process (linearly) is regular in accordance with the presented in §1 chapter IV results:

a) it has spectral density f (λ), representable in the form $f = |g|^2$, where g is an external function of class $\mathcal{H}^2$ in the upper half-plane;

b)
$$\int_{-\infty}^{\infty} \frac{\ln f(\lambda)}{1+\lambda^2}\, d\lambda < \infty.$$

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Hence and from theorem 2 chapters II immediately follow that the coefficient of regularity ρ (τ) (equal regarding $\sup |M\eta_1\bar{\eta}_2|$, where $\eta_1 \in H(-\infty, 0)$, $\eta_2 \in H(0, \infty)$, $M|\eta_1|^2 = M|\eta_2|^2 = 1$)) has the following analytical expression:

$$\rho(\tau) = \sup_{\varphi,\,\psi} \left| \int_{-\infty}^{\infty} e^{i\tau\lambda} \varphi(\lambda)\, \psi(\lambda)\, f(\lambda)\, d\lambda \right|, \qquad (1.1)$$

where sup it is taken on all φ, ψ of the single sphere $E^+(f)$ of space $L^+(f) = 1/g\, \mathcal{H}^2$. (Here, as in the previous chapter, we will deal only with absolutely continuous spectral functions F, G, ... and therefore instead of $L^+(F)$, $L^+(G)$, $\|\cdot\|_F$, $(\ldots)_G$ etc. we prefer to write $L^+(f)$, $L^+(g)$, $\|\cdot\|_f$, $(\ldots)_g$, where $f = F'$, $g = G'$, ...).

It is understandable that value ρ (r) will not be changed, if we in (1.1) take sup not in the entire sphere $E^{+}(f)$, but according to any of its dense subset, for example on $\mathscr{H}^2 \cap E^{+}(f)$, according to the functions of form $\sum a_l e^{i\lambda t_l}$, $t_l \geqslant 0$, and so forth.

Analogous with equality (1.5) chapter V, from (1.1) it is possible to deduce the following equality:

$$\rho(\tau)=\sup_{\theta}\left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\theta(\lambda) f(\lambda)\, d\lambda\right|, \qquad (1.2)$$

where sup it is taken on all $\theta \in \mathscr{H}^1$, $\|\theta\|_f^{(1)} \leqslant 1$.

Finally, by using by equality $L^{+}(f)=\frac{1}{g}\mathscr{H}^2$ and by substituting in (1.1) φ, ψ to φ_1/g, ψ_1/g, $\varphi, \psi \in \mathscr{H}^2$, let us find

$$\rho(\tau)=\sup_{\varphi_1, \psi_1}\left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\varphi_1(\lambda)\, \psi(\lambda)\, \frac{\overline{g(\lambda)}}{g(\lambda)}\, d\lambda\right|, \qquad (1.3)$$

where now φ_1, ψ_1 they pass single sphere $\mathscr{H}^2$. This equality is analogous with formula (1.6) chapter V.

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From (1.3) also it follows (comp. conclusion (1.7) from (1.6) in chapter V), that

$$\rho(\tau)=\sup_{\theta}\left|\int_{-\infty}^{\infty} \theta(\lambda)\, e^{i\lambda\tau}\, \frac{\overline{g(\lambda)}}{g(\lambda)}\, d\lambda\right|, \quad \theta \in \mathscr{H}^1, \quad (1.4)$$

$$\|\theta\|^{(1)} \leqslant 1.$$

Further, considering integral in the right side of the last/latter equality as functional $l(\theta)$ in $\mathcal{L}^1$, possible, in the same way as this was done in §2 chapter V, to arrive at the equality

$$\rho(\tau)=\inf_A \left\| \frac{\bar{g}}{g} e^{i\lambda\tau} - A \right\|,$$

where inf it is taken on all $A \in \mathcal{H}^\infty$ in the upper half-plane. Being transmitted of this equality, it is possible to move on the way, indicated in §2 chapter V, almost to the end itself; the obstructions, which do not make it possible to obtain so final a result as theorem 3 chapters V, they will appear only at the very end of the proof, when instead of polynomials P (see page 213) they will arise the integral functions of the final degree and it will necessary prove their independence of ε. Therefore we will be bounded to the following result, which, just as theorem 1 chapter V, has as a goal to give the preliminary representations of the spectral densities of completely regular processes.

Theorem 1. Let spectral density f (λ) of stationary process $\xi(t)$ take the form

$$f(\lambda)=|\Gamma(\lambda)|^2 w(\lambda), \qquad (1.5)$$

where Γ (λ) - the summarized with square integral function of the final degree, and function w with all $\varepsilon>0$ it allow/assumes the representation

$$w=\exp\{r_\varepsilon+t_\varepsilon+\tilde{s}_\varepsilon\}, \qquad (1.6)$$

where r_ε - is the limited real evenly continuous on (-∞, ∞) function, is real function with $(-\infty, \infty)$ and t_ε - is the function, $\|t_\varepsilon\|^{(\infty)} \leqslant \varepsilon$ и $\tilde{s}_\varepsilon$ - conjugate/combined (harmonically) function s_ε with $\|s_\varepsilon\|^\infty \leqslant \varepsilon$. In that case the process $\xi(t)$ is completely regular.

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Let us explain first, which is understood by the function, conjugate/combined to bounded function. Let us designate by W^2 the class of the measurable functions α (λ), for which $\frac{a}{1+|\lambda|} \in \mathscr{L}^2(-\infty, \infty)$. Specifically, $\mathscr{L}^\infty \subset W^2$. Let us register

$$a(\lambda) = (\lambda - i)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \alpha(u)\, e^{i\lambda u}\, du,$$

where α (u) - Fourier transform function $\frac{a(\lambda)}{\lambda - i} \in \mathscr{L}^2$. Adjoint function $\tilde{\alpha}$ let us determine by equality

$$\tilde{a}(\lambda) = (\lambda - i)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \alpha(u)\, i\, \mathrm{sign}\, u \cdot e^{i\lambda u}\, du \qquad (1.7)$$

(for greater detail, see [1], by page 171). From (1.7) it follows that

$$\int_{-\infty}^{\infty} \frac{|\tilde{a}(\lambda)|}{1+\lambda^2}\, d\lambda \leqslant \frac{1}{\sqrt{2\pi}} \|(\lambda + i)^{-1}\|^{(2)} \cdot \|\alpha\|^{(2)}. \qquad (1.8)$$

In the proof of theorem 1 and below we will meet the nonnegative functions f, which can turn out to be those which are nonsummarized, but for which all the same the determined by equality (1.1) value $\rho(\tau) \longrightarrow 0$. In connection with this by us will be required the following class of the functions, which we will call completely regular.

Determination. the nonnegative measurable function f (λ) will be called completely regular, if

$$1)\ \int_{-\infty}^{\infty} \frac{\ln}{1+\lambda^2}\, d\lambda < \infty;$$

$$2)\ \rho(\tau) = \sup_{\varphi,\psi} \left| \int_{-\infty}^{\infty} \varphi(\lambda)\, \psi(\lambda)\, e^{i\lambda\tau} f(\lambda)\, d\lambda \right| \xrightarrow[\tau\to\infty]{} 0,$$

whereupon sup are taken on all ϕ, $\psi \in \mathcal{H}^{2+}$ with $\|\varphi\|_f = \|\psi\|_f = 1$.

Any completely regular summarized function is the spectral density of completely regular process, and, on the contrary, the spectral density completely of regular process to eat a completely regular function.

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On the strength of requirement 1) any completely regular

function is factorable: f (λ) = g $|(\lambda)|^2$, where

$$g(z)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1+\lambda z}{\lambda-z}\,\frac{\ln f(\lambda)}{1+\lambda^2}\,d\lambda\right\}$$

is an external function in the upper half-plane. Hence immediately it follows (comp. page 60) [1], that closing/shorting $L^+(f)$ in L (f) of set $\mathscr{H}^2\cap L(f)$ is equal to $\frac{1}{g}\mathscr{H}^2$.

FOOTNOTE [1]. Here is proof. Function $w^+=\max(1, w)$ is factorable: $w^+=|g^+|^2$, where g^+ is an external function. If ϕ - external function from $\mathscr{H}^2$, $\varphi/g^+\in\mathscr{H}^2$ and $\varphi/g^+\in L(w)$, so that $\mathscr{H}^2\cap L(w)$ is not empty. Further ϕg - external function from $\mathscr{H}^2$, and according to the theorem of Lax (see §1 chapter II) in $g\varphi\mathscr{H}^2$, a that means in $g\mathscr{H}^2$ is contained the set, dense in $\mathscr{H}^2$. Consequently, $\frac{1}{g}\mathscr{H}^2\subset L^+(\omega)$. On the contrary, if $\psi\in L^+(w)$, that is function ψg, being the limit in $\mathscr{L}^2$ of functions $\varphi_n g$, $\varphi_n\in\mathscr{H}^2$, $\varphi_n g\in\mathscr{H}^2$, itself is a function from $\mathscr{H}^2$, i.e. $L^+(w)\subset\frac{1}{g}\mathscr{H}^2$. ENDFOOTNOTE.

Returning to theorem 1, let us demonstrate first that function w, which allow/assumes representation (1.6), is completely regular. The evenly continuous bounded function e^{r_ε} it is possible as conveniently well to approach by the limited integral functions of the final stage (see [25]). Let Q_T - be the limited integral function of degree $\leqslant$T, for which

$$e^{r_\varepsilon}=Q_T(1+\theta),\quad |\theta|\leqslant\varepsilon.$$

Then

$$w(\lambda) = Q_T e^{\tilde{v}_\varepsilon - iv_\varepsilon}(1+\theta_\varepsilon), \quad \|\theta_\varepsilon\|^{(\infty)} \leqslant 7\varepsilon.$$

Let us assume $w_\varepsilon = |Q_T||e^{\tilde{v}_\varepsilon - iv_\varepsilon}|$. If ε is sufficiently small, then for all $\varphi \in L(w)$

$$\frac{1}{2}\|\varphi\|_{w_\varepsilon} \leqslant \|\varphi\|_w \leqslant 2\|\varphi\|_{w_\varepsilon}.$$

Further, on (1.8) $\frac{\tilde{v}_\varepsilon}{1+\lambda^2} \in \mathscr{L}^1$, so that $L^+(w) = L^+(e^{\tilde{v}_\varepsilon}) = \frac{1}{\gamma}\mathscr{H}^2$, where the external function $\gamma(z)$ is determined by equality $|\gamma(\lambda)|^2 = e^{\tilde{v}_\varepsilon}$. It is obvious, $\gamma = \exp\left\{\frac{1}{2}(\tilde{v}_\varepsilon - iv)\right\}$.

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If ϕ, $\psi \in L^+(w)$, that $\phi = \phi_1/\gamma$, $= \psi = \psi_1/\gamma$, where ϕ_1, $\psi_1 \in \mathscr{H}^2$. Furthermore, for $\tau > T$ $e^{i\lambda\tau}Q_T \in \mathscr{H}^\infty$, so that, for example, $e^{i\lambda\tau}\varphi_1 Q_T \in \mathscr{H}^2$. Consequently, for all $\tau > T$ and all ϕ, $\psi \in L^+(w)$

$$\int_{-\infty}^{\infty} \varphi(\lambda)\psi(\lambda)e^{i\lambda\tau}Q_T(\lambda)e^{\tilde{v}_\varepsilon - iv_\varepsilon}\,d\lambda = \\ = \int_{-\infty}^{\infty} [\varphi_1(\lambda)e^{i\lambda\tau}Q_T(\lambda)]\,\psi_1(\lambda)\,d\lambda = 0.$$

Therefore for all $\tau > T$

$$\rho(\tau, w) = \sup_{\varphi,\psi}\left|\int_{-\infty}^{\infty}\varphi(\lambda)\psi(\lambda)e^{i\lambda\tau}w(\lambda)\,d\lambda\right| \leqslant \\ \leqslant \sup_{\varphi,\psi}\int_{-\infty}^{\infty}|\varphi(\lambda)||\psi(\lambda)||w_\varepsilon(\lambda)||\theta_\varepsilon(\lambda)|\,d\lambda \leqslant 28\varepsilon.$$

Now the assertion of theorem ensues from the following lemma, which is the analog of theorem 2 chapters V.

Lemma 1. If function w completely is regular, Γ - limited integral function of the final degree $\leq T$, a $f = |\Gamma|^2 w$, then is function f also completely is regular, whereupon

$$\rho(\tau; f) \leqslant \rho(\tau - 2T; w). \qquad (1.9)$$

Really/actually $\frac{\ln|\Gamma|}{1+\lambda^2} \in \mathscr{L}^1$ and in $\tau \geqslant T$ both functions are such, that $e^{i\lambda\tau}\Gamma,\ e^{\lambda i\tau}\overline{\Gamma} \in \mathscr{H}^{\infty}$ [1].

FOOTNOTE [1]. recall that $\overline{\Gamma}(z) = \overline{\Gamma(\bar{z})}$ ENDFOOTNOTE.

Therefore, if φ, ψ belong to the single sphere $L^+(f)$, then $e^{i\lambda T}\Gamma\varphi,\ e^{i\lambda T}\overline{\Gamma}\psi$ they belong to the single sphere of space $L^+(w)$, and, which means,

$$\rho(\tau; f) = \sup_{\varphi,\psi}\left|\int_{-\infty}^{\infty}\varphi(\lambda)\psi(\lambda)e^{i\lambda\tau}w(\lambda)|\Gamma(\lambda)|^2\,d\lambda\right| =$$

$$= \sup_{\varphi,\psi}\left|\int_{-\infty}^{\infty}[\varphi e^{i\lambda T}\Gamma][\psi e^{i\lambda T}\overline{\Gamma}]e^{i\lambda(\tau-2T)}w\,d\lambda\right| \leqslant \rho(\tau - 2T; w).$$

Lemma, and with it and theorem 1 are demonstrated.

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Corollary. If spectral density takes form $f(\lambda)=|\Gamma(\lambda)|^2 w(\lambda)$, where Γ - the summarized with square integral function of degree $\leq T$, and function w is limited on top, also, from below ($0 < m \leqslant w \leqslant M < \infty$) and evenly continuous on $(-\infty,\infty)$, then is process $\xi(t)$ is completely regular, whereupon

$$\rho(\tau)\leqslant\frac{1}{m}A_{\tau-2T}(\omega). \qquad (1.10)$$

Here and throughout through $A_\sigma(h)$ is designated the value of the best uniform approximation of function h by the integral functions of the final degree $\leq\sigma$.

Actually, $w=e^{\ln w}$ and under conditions of corollary function lnw is limited and evenly continuous. Inequality (1.10) easily follows from the computations, used in the proof of theorem 1.

§2. Research of special function γ (T; μ).

In this and two following paragraphs we will spread to the continuous case results §§3, 4 chapters V. As there, we will begin from the study of the behavior of Fejer's integral

$$\gamma(T;\mu)=\int_{-\infty}^{\infty}\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}f(\lambda+\mu)\,d\lambda \qquad (2.1)$$

with Vol. $T\to\infty$. Here f (λ) - the spectral density of completely regular process $\xi(t)$. Subsequently function γ (T; μ) plays the same role, as function γ (N; μ) chapter V.

Let us note that

$$\gamma(T;\mu)=\mathsf{M}\left|\int_0^T e^{-i\mu t}\xi(t)\,dt\right|^2. \qquad (2.2)$$

For this proof let us pass to the spectral representation of the process

$$\xi(t)=\int_{-\infty}^{\infty}e^{it\lambda}\Phi(d\lambda),$$

where $\Phi(d\lambda)$ - orthogonal random measure (M $|\Phi(d\lambda)|^2$ = f (λ) dλ.

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We will have

$$M\left|\int_0^T e^{-i\mu t}\xi(t)\,dt\right|^2 = M\left|\int_{-\infty}^{\infty}\frac{e^{i(\lambda-\mu)T}-1}{i(\lambda-\mu)}\Phi(d\lambda)\right|^2 =$$
$$= \int_{-\infty}^{\infty}\frac{\sin^2 T\frac{\lambda-\mu}{2}}{(\lambda-\mu)^2}f(\lambda)\,d\lambda = \int_{-\infty}^{\infty}\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}f(\lambda+\mu)\,d\lambda. \quad (2.3)$$

Lemma 2. If stationary process $\xi(t)$ has spectral density f (λ), then at Vol. $T\to\infty$ value

$$M\left|\int_0^T \xi(t)\,dt\right|^2 = \int_{-\infty}^{\infty}\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}f(\lambda)\,d\lambda$$

either approaches ∞ or is limited. The latter occurs in that and only that case when $\int_{-\infty}^{\infty}\frac{f(\lambda)}{\lambda^2}d\lambda<\infty$. The proof of this and two following lemmas very is similar to the proof of analogous assertions (lemma 4-6) chapter V. Therefore we will give the detailed proof only of last/latter lemma 4.

Lemma 3. With each μ, when T ⟶ ∞, function γ (t) either approaches ∞ or is limited. The latter occurs in that and only that case, when $\int_{-\infty}^{\infty}\frac{f(\lambda)}{(\lambda-\mu)^2}d\lambda<\infty.$

Lemma 4. Let 0 < a < ∞. With Vol. $T\to\infty$ or inf γ (T; μ) $\to\infty$,

or there is a point $\theta \in [-a, a]$ $|\mu| \leq a$ such, that $\int\limits_{-\infty}^{\infty} \frac{f(\lambda)}{(\lambda - \theta)^2} d\lambda < \infty.$

Proof in many respects is analogous with the discrete case (lemma 6 chapters V).

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Specifically, equality $U^\tau \xi(t) = \xi(t+\tau)$ it determines in hilbert space H - (- ∞, ∞) the group of the unitary operators $\{U^\tau\}$. If

$$\varliminf_{T \to \infty} \inf_{|\mu| \leq a} \gamma(T; \mu) < \infty,$$

we, discussing then just as in the proof of lemma 6 chapters V, determine the sequence of the cell/elements

$$\zeta_k \in H(-\infty, \infty), \quad \zeta_k = \int\limits_0^{T_k} e^{-it\theta_k} U^t \xi(0)\, dt,$$

weakly converging to certain cell/element $\zeta \in H(-\infty, \infty)$. In this case numerical sequence θ_k converge to certain number $\theta \in [-a, a]$. Consequently, $\lim\limits_{(\text{сл.})} e^{-i\tau\theta_k} \zeta_k = e^{-i\tau\theta}\zeta;$ here $\lim\limits_{(\text{сл.})}$ indicates weak limit, a τ - arbitrary real number.

According to Riemann - Lebesgue's theorem

$$B(s) = \mathsf{M}\xi(t+s)\overline{\xi(t)} = (\xi(t+s), \xi(t)) =$$
$$= \int\limits_{-\infty}^{\infty} e^{i\lambda s} f(\lambda)\, d\lambda \xrightarrow[|s| \to \infty]{} 0.$$

Therefore with all $h \in H(-\infty, \infty)$

$$\lim_{s\to\infty} (\xi(s), h) = 0, \quad \lim_{(\text{сл.})\, s\to\infty} \xi(s) = 0,$$

and, therefore,

$$\zeta - e^{-i0\tau} U^\tau \zeta = \lim_{(\text{сл.})} (\zeta_k - e^{-i0_k\tau} U^\tau \zeta_k) = \\ = \lim_{(\text{сл.})} \left(\int_0^\tau e^{-i0_k t} \xi(t)\, dt + \int_{T_k}^{T_k+\tau} e^{-i0_k t} \xi(t)\, dt \right) = \\ = \int_0^\tau e^{-i0t} \xi(t)\, dt. \quad (2.4)$$

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We form stationary process $\eta(t) = e^{-i0t} U^t \zeta$. On the strength of (2.4)

$$\eta(t) - \eta(t+\tau) = e^{-i0t} U^t (\zeta - e^{-i0\tau} U^\tau \zeta) = \int_t^{t+\tau} e^{-i0s} \xi(s)\, ds. \quad (2.5)$$

The last/latter equality means that process $\eta(t)$ we differentiate on the average quadratic and that

$$\eta'(t) = -e^{-i0t} \xi(t). \quad (2.6)$$

Let us designate through $f_\eta(\lambda), f_\xi(\lambda)$ the spectral densities of

processes $\eta(t)$, $\xi(t)$. From (2.6) escape/ensues the following relationship/ratio between f_ξ, f_η:

$$\lambda^2 f_\eta(\lambda) = f_\xi(\lambda + \theta).$$

Consequently,

$$\int_{-\infty}^{\infty} \frac{f(\lambda)\, d\lambda}{(\lambda-\theta)^2} = \int_{-\infty}^{\infty} f_\eta(\lambda)\, d\lambda < \infty.$$

On the contrary, if $\int_{-\infty}^{\infty} \frac{f(\lambda)}{(\lambda-\theta)^2}\, d\lambda < \infty$, that on lemma 3 $\overline{\lim\limits_{T\to\infty}}\, \gamma(T;\ \theta) < \infty$. Lemma 4 is demonstrated.

Lemma 5. Let process $\xi(t)$ completely regular. Will be located by integrating with square integral function of the final degree $\Gamma(\lambda)$ such, that

$$\int_{-\infty}^{\infty} \frac{|\Gamma(\lambda)|^2}{f(\lambda)}\, d\lambda < \infty. \qquad (2.7)$$

Proof literally repeats the first part of the proof of lemma 7 chapters V; only now necessary to use results about the "noninterpolatability" of processes with continuous time [1]).

FOOTNOTE [1]. See, for example, [22], page 183. ENDFOOTNOTE.

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Lemma 6. Let process $\xi(t)$ completely regular. Whatever number a, 0 < a < ∞, spectral density f (λ) of process $\xi(t)$ it is possible to present in the form

$$f(\lambda) = w(\lambda)\,|P_a(\lambda)|^2, \qquad (2.8)$$

where $P_a(\lambda)$ – algebraic polynomial with real roots (arrange/located within interval [-a, a]), a w (λ) it possesses those by property, that

$$\lim_{T}\ \inf_{|\mu|\leqslant a} \int_{-\infty}^{\infty} w(\lambda)\, \frac{\sin^2 T\,\frac{\lambda-\mu}{2}}{(\lambda-\mu)^2}\, d\lambda = \infty. \qquad (2.9)$$

Proof. Let us designate by $\mathscr{P}$ the set of all polynomials Q (λ) with roots inside [-a, a], for which

$$\int_{-a}^{a} \frac{|Q(\lambda)|^2}{f(\lambda)}\, d\lambda < \infty.$$

In accordance with the previous lemma the set $\mathscr{P}$ is not empty. (As Q (λ) it is possible to take the polynomial whose roots coincide with roots Γ (λ), that lie on [-a, a]. Let $Q_a(\lambda)$ – that of the polynomials $Q(\lambda) \in \mathscr{P}$, which have a smallest degree and coefficient 1 with leading term. Among polynomicals P (λ) with roots inside [-a, a], for which $\int_{-\infty}^{\infty} \frac{f(\lambda)}{|P(\lambda)|^2}\, d\lambda < \infty$, there is a polynomial $P_a(\lambda)$ the maximum (final) degree. Actually, from the inequality

$$\left(\int_{-a}^{a} \left|\frac{Q_a(\lambda)}{P(\lambda)}\right| d\lambda\right)^2 \leqslant \int_{-a}^{a} \frac{|Q_a(\lambda)|^2}{f(\lambda)}\, d\lambda \cdot \int_{-a}^{a} \frac{f(\lambda)}{|P(\lambda)|^2}\, d\lambda$$

it follows that all polynomials P (λ) divide polynomial $Q_a(\lambda)$.

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Let us place now f (λ) = $|P_a(\lambda)|^2 w(\lambda)$. It is obvious,

$$\int_{-\infty}^{\infty} w(\lambda)\, d\lambda < \infty,$$

so that w (λ) there is the spectral density of certain stationary process. Further reasonings coincide with the reasonings, used in the proof of lemma 7 chapters V, and we will drop/omit them. Lemma is demonstrated.

Lemma 7. Let process $\xi(t)$ completely regular. At those points μ, where $\lim_T \gamma(T;\ \mu) = \infty$, function γ (T; μ) is represented in the form γ (T: μ) = Th (T: μ), where h (T; μ) is slowly varying function T, i.e., with all x > 0

$$\lim_{T\to\infty} \frac{h(Tx;\ \mu)}{h(T;\ \mu)} = 1.$$

Proof to a considerable degree is similar to the proof of the analogous assertions of chapter V (lemma 8-10), and therefore we will drop/omit some parts. It is necessary to show that with all x > 0

$$\lim_T \frac{\gamma(Tx)}{\gamma(T)} = x \qquad (2.10)$$

(here below we for a brevity lower argument μ). as in chapter V, we first will disassemble the case whole x, then will establish/install some properties of function γ (t), after which will demonstrate (2.10) for all x.

1. x are integer, x = k. Let us register γ (kT) in the form

$$\gamma(kT) = \mathsf{M}|z_1 + y_1 + \ldots + z_k + y_k|^2,$$

where

$$z_j = \int\limits_{(j-1)T+(j-1)r}^{jT+(j-1)r} e^{-i\mu t}\xi(t)\,dt, \quad j = \overline{1, k},$$

$$y_j = \int\limits_{jT+(j-1)r}^{jT+jr} e^{-i\mu t}\xi(t)\,dt, \quad j = \overline{1, k-1},$$

$$y_k = -\int\limits_{kT}^{kT+(k-1)r} e^{-i\mu t}\xi(t)\,dt,$$

a r = r (T) selected so [1], in order to r (T) $\underset{T\to\infty}{\to}\infty$, but

$$\frac{\gamma(r)}{\gamma(T)} \xrightarrow[T\to\infty]{} 0, \quad \max_{1\leqslant s\leqslant k} \frac{\gamma(sr)}{\gamma(T)} \xrightarrow[T\to 0]{} 0.$$

FOOTNOTE [1]. Since $\gamma(T) = \int\limits_{-\infty}^{\infty} \frac{\sin^2\frac{T\lambda}{2}}{\lambda^2} f(\lambda+\mu)\,d\lambda \leqslant T^2 \int\limits_{-\infty}^{\infty} f(\lambda)\,d\lambda.$

it is possible to place r = ln γ (t), since then $\gamma(sr) \leqslant k^2 \ln^2 \gamma(T) \int_{-\infty}^{\infty} f(\lambda)\, d\lambda.$

ENDFOOTNOTE.

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It is not difficult to count, that

$$\mathsf{M}|z_j|^2 = \gamma(T); \quad \mathsf{M}|y_i|^2 = \gamma(r), \quad j \leqslant k-1;$$
$$\mathsf{M}|y_k|^2 = \gamma((k-1)r).$$

Therefore the same reasonings, as on page 225, they make it possible to claim the following:

$$\gamma(kT) = k\gamma(T)\left(1 + o\left(k\rho(r) + k\left(\frac{\gamma(r)}{\gamma(T)}\right)^{1/2} + k\left(\frac{\gamma((k+1)r)}{\gamma(T)}\right)^{1/2}\right)\right) = k\gamma(T)(1 + o(1)).$$

The case whole x is dismantle/selected.

2. x is rational, x = p/q. We have

$$\lim_T \frac{\gamma\left(\frac{p}{q}T\right)}{\gamma(T)} = \lim_T \frac{\gamma\left(p\frac{T}{q}\right)}{\gamma\left(\frac{T}{q}\right)} \frac{\gamma\left(\frac{T}{q}\right)}{\gamma\left(q\frac{T}{q}\right)} = \frac{p}{q}.$$

3. Arbitrary x. Just as on page 225, it is derive/concluded, that

$$\ln h(T) = o(\ln T)$$

and, therefore, with all $\varepsilon > 0$

$$\lim_{T} T^{\varepsilon} h(T) = \infty, \quad \lim_{T} T^{-\varepsilon} h(T) = 0. \qquad (2.11)$$

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Being based on equalities (2.11) and discussing just as on page 228, it is not difficult to show that the functions $\psi_1(x)$, $\psi_2(x)$, determined by the relationship/ratios

$$\psi_1(x) = \underline{\lim}_{T} \frac{\gamma(Tx)}{\gamma(T)}, \quad \psi_2(x) = \overline{\lim}_{T} \frac{\gamma(Tx)}{\gamma(T)},$$

are continuous. Since for rational x ψ_1 (x) = ψ_2 (x) = x, also for all real x

$$\lim_{T} \frac{\gamma(Tx)}{\gamma(T)} = x.$$

Lemma is demonstrated.

Lemma 8. If $\gamma_a(T) = \inf_{|\mu| \leqslant a} \gamma(T; \mu) \underset{T \to \infty}{\to} \infty$, that relationship/ratio

$$\lim_{T} \frac{h(Tx)}{h(T)} = 1 \qquad (2.12)$$

occurs evenly on all μ and x as such, that $\mu < a$, $0 < x_0 < x < x_1 < \infty$.

This lemma is proven literally just as lemma 10 chapters V; therefore we will drop/omit proof.

§3. Proof of the fundamental theorem about the necessary conditions.

The target/purpose of this paragraph is proof of theorem 2, analogous to theorem 5 chapters V.

Theorem 2. Let $\xi(t)$ – be a completely regular process with spectral density f (λ). Whatever number a, $0 < a < \infty$, function f (λ) is representable in the form

$$f(\lambda) = |P_a(\lambda)|^2 w_a(\lambda),$$

where $P_a(\lambda)$ – algebraic polynomial with roots on [-a, a] a original $W_a(\lambda)$ functions $w_a(\lambda)$ satisfies the condition

$$\omega_W(\delta) = \sup_{|\lambda| \leq a} \sup_{|t| \leq \delta} \frac{|W_a(\lambda+t) + W_a(\lambda-t) - 2W_a(\lambda)|}{|W_a(\lambda+t) - W_a(\lambda-t)|} \xrightarrow[\delta \to 0]{} 0. \quad (3.1)$$

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This condition is a full/total/complete analog of condition (3.2) chapter V; it assigns on smoothness and order of zero spectral densities f (λ) is accurate the same limitations, as condition (3.2)

chapter V in the discrete case. Actually, by the same methods, as on page 249, from (3.1) is derive/concluded

Corollary 1. The spectral density of completely regular process does not have first-order discontinuities.

Small further reasonings will be required in order to demonstrate.

Corollary 2. The lower order of any zero λ_0 the spectral density $f(\lambda)$ of completely of regular process directly is equal either zero or to the whole even number. Consequently, the present order of any zero $f(\lambda)$ can be only whole and even.

Corollary 3. For all $\varepsilon > 0$

$$\varlimsup_{\lambda \to \lambda_0} f(\lambda) |\lambda - \lambda_0|^{\varepsilon} = 0.$$

We will begin from the proof of the following assertion:

if carried out condition (3.1), then all the functions

$$h_{\mu}(x) = h(x) = \frac{1}{x} \int_0^x w_a(\lambda + \mu)\, d\lambda$$

are slowly varying with $x \longrightarrow 0$, whereupon $\lim_{n \to 0} \frac{h(zx)}{h(x)} = 1$ evenly on all $\mu \in [-a, a]$ и $z \in [s, S]$, $0 < s < S < \infty$.

Actually, if z are whole, then it is obvious, (3.1) it gives

$$\frac{\frac{1}{zx} h(zx)}{\frac{1}{x} h(x)} = 1 + o(1).$$

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Hence easily it follows that $\lim \frac{h(zx)}{h(x)} = 1$ for all rational z. The final part of the proof coincides with the termination of the proof of lemmas 7 and 8; in this case one should still use the following obvious inequality: for $\varepsilon < 1/q$

$$\int_0^{\varepsilon x} w_a(\lambda + \mu)\, d\lambda \leqslant \int_0^{x/q} w_a(\lambda + \mu)\, d\lambda.$$

Let us demonstrate now that the lower order $(\underline{k}(\lambda_0))$ any zero λ_0 function $w_a(\lambda)$ of theorem 2 is equal to zero. If $\underline{k}$ (λ_0) > $\varepsilon > 0$, that in certain vicinity of point λ_0 $w_a(\lambda) < |\lambda - \lambda_0|^{\varepsilon/2}$ and, which means,

$$h_{\lambda_0}(x) = \frac{1}{x} \int_0^x w_a(\lambda + \lambda_0)\, dx < x^{\varepsilon/2},$$

but this is impossible, since function h (x) - slowly varying and with all $\varepsilon > 0$ $\lim_{x \to 0} h(x)\, x^{-\varepsilon} = \infty$.

Corollary 2 is proved. Analogously is proven corollary 3.

Let us pass to the proof of theorem 2. It is based on the same

ideas, as the proof of theorem 5 chapters is V. Just as the mentioned result, theorem 2 is a proposition is Tauberian the type, in which on the basis of the properties of the function

$$\gamma(T;\ \mu)=\int_{-\infty}^{\infty}\frac{\sin^2 T\frac{\lambda-\mu}{2}}{(\lambda-\mu)^2}\,dW_a(\lambda)$$

it is discussed the local behavior $W_a(\lambda)$.

Preliminarily, relying on lemma 6 and following below lemma 9, let us get rid from zero f (λ).

Lemma 9. If f (λ) - the spectral density of completely regular process $\xi(t)$, all poles even zero rational integral function $R(\lambda)=\frac{P(\lambda)}{Q(\lambda)}$ are real, but function $w=|R|^2 f\in\mathscr{L}^1(-\infty,\ \infty)$, that w (λ) is also the spectral density of stationary process $\eta(t)$, whereupon

$$\rho(\tau;\ w)\leqslant\rho(\tau;\ f).$$

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Proof. If function φ, ψ belongs to the single sphere of space $L^+(w)$, that φ/R, $\psi/\bar R$ they lie/rest at the single sphere L^+ (f). Therefore

$$\rho(\tau;\ w)=\sup_{\varphi,\psi}\left|\int_{-\infty}^{\infty}\varphi(\lambda)\psi(\lambda)e^{i\lambda\tau}w(\lambda)\,d\lambda\right|=$$

$$=\sup_{\varphi,\psi}\left|\int_{-\infty}^{\infty}\frac{\varphi(\lambda)}{R(\lambda)}\frac{\psi(\lambda)}{\bar R(\lambda)}e^{i\lambda\tau}f(\lambda)\,d\lambda\right|\leqslant\rho(\tau;\ f).$$

By entering as in §4 chapter V, let us (by beginning with this place up to the termination of the proof of lemma 12) consider that f (λ) there is the spectral density of the completely regular sequence, for which

$$\lim_{T} \inf_{|\mu| \leq a} \gamma(T;\ \mu) = \infty.$$

Lemma 10. Let a (λ) - the even function with the limited third derivative, which turns into zero outside interval [-1, 1]. Then with Vol. $T \to \infty$ is evenly on $|\mu| \leqslant$ a

$$\frac{1}{T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{T\lambda}{2}}{\lambda^2} f(\lambda + \mu)\, a(T\lambda)\, d\lambda = \frac{1}{\pi} \int_{0}^{1} \frac{\sin^2 \frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2} a(\lambda)\, d\lambda \cdot h(T)(1 + o(1)), \quad (3.2)$$

where for brevity h (T; μ) it is designated through h (T).

Proof, in fact, coincides with the proof of lemma 12 chapters V.

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Let us introduce Fourier transform A (z) of function a (λ) and according to that which was assigned $\varepsilon>0$ let us define numbers s, δ and set of $\mathcal{B}$ just as on page 231. Then let us rewrite left side (3.2) in the form

$$\frac{1}{\sqrt{2\pi}}\left[\int_{1+\delta}^{s} z\,[h(T(z+1))+h(T(z-1))-2h(Tz)]\,A(z)\,dz+\right.$$
$$+\int_{1+\delta}^{s}[h(T(z+1))-h(T(z-1))]\,A(z)\,dz+$$
$$+\int_{\delta}^{1-\delta} z\,[h(T(z+1))-h(T(z-1))-2h(Tz)]\,A(z)\,dz+$$
$$+\int_{\delta}^{1-\delta}[h(T(z+1))+h(T(z-1))]\,A(z)\,dz+R(T),$$

where R $|(T)|\leqslant \varepsilon h(T)$.

Hence and from lemma 8 it is derive/concluded already, which with T ⟶ ∞ is evenly on $|\mu|\leqslant a$

$$\frac{1}{T}\int_{-\infty}^{\infty}\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}f(\lambda+\mu)\,a(T\lambda)\,d\lambda=$$
$$=\sqrt{\frac{2}{\pi}}\int_{0}^{1}(1-z)\,A(z)\,dz\cdot h(T)\,(1+o(1))=$$
$$=\frac{1}{\pi}\int_{0}^{1}\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}a(\lambda)\,d\lambda\,(1+o(1)). \quad (3.3)$$

Lemma is demonstrated.

Lemma 11. Let α (λ) be the odd, three times differentiated function with limited third derivative, which turns into zero not interval [-1, 1]. Then with $T \longrightarrow \infty$ is evenly on $|\mu| \leqslant a$

$$\frac{1}{T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{T\lambda}{2}}{\lambda^2} f(\lambda+\mu)\, a(T\lambda)\, d\lambda = o(h(T)), \qquad (3.4)$$

where as before h (T) = h (T; μ).

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The proof of this lemma even is somewhat simpler than the proof of its discrete analog (lemma 13 chapters V). Having again designated by A (z) Fourier transform function a (λ), let us rewrite being subject to estimation integral in the form

$$\frac{1}{T} \int_{B} A(z)\, dz \int_{-\infty}^{\infty} \frac{\sin^2 \frac{T\lambda}{2}}{\lambda^2} \sin T\lambda z f(\lambda+\mu)\, d\lambda + R(T),$$

where $|R(T)| \leqslant eh(T)$, and B is a set $\{ z: \delta < z < 1 - \delta, 1 + \delta < z < s\}$

It is not difficult to count, that

$$\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}=\frac{1}{4}\left|\int_0^T e^{i\lambda\tau}\,d\tau\right|^2.$$

Further,

$$\sin T\lambda z=\frac{1}{2i}\left(e^{iT\lambda z}-e^{-iT\lambda z}\right),$$

so that for those z, for which Tz > T + T_0, the products $\exp\{iT\lambda z\}$, $\exp\{-iT\lambda z\}$ on $\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}$ will be registered respectively in the form

$$e^{iT_0\lambda}\varphi_z^+(\lambda),\quad e^{-iT_0\lambda}\varphi_z^-(\lambda),$$

where $\varphi_z^+(\lambda)$, $\varphi_z^-(\lambda)$ they belong with respect to spaces $\mathscr{H}^1$ in the upper and lower half-plane. On the strength of full/total/complete regularity [1])

$$\frac{1}{T}\left|\int_{B\cap(Tz>T+T_0)} A(z)\,dz\int_{-\infty}^{\infty}\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}\sin T\lambda z f(\lambda+\mu)\,d\lambda\right|=$$
$$=\frac{1}{T}\left|\int_{B\cap(Tz>T+T_0)} A(z)\,dz\int_{-\infty}^{\infty}\left[e^{iT_0\lambda}\varphi_z^+(\lambda)+e^{-iT_0\lambda}\varphi_z^-(\lambda)\right]\times\right.$$
$$\left.\times f(\lambda+\mu)\,d\lambda\right|\leqslant\rho(T_0)\int_{-\infty}^{\infty}|A(z)|\,dz\cdot\frac{1}{T}\int_{-\infty}^{\infty}\left(|\varphi_z^+(\lambda)|+\right.$$
$$\left.+|\varphi_z^-(\lambda)|\right)f(\lambda+\mu)\,d\lambda\leqslant\varepsilon h(T),\qquad(3.5)$$

if only T_0 it is selected by sufficiently large.

FOOTNOTE [1]. Together with f (λ) with all μ f (μ + λ) = $f_\mu(\lambda)$ are the

spectral density of completely regular process with those coefficient of mixing. ENDFOOTNOTE.

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Thus, to us remained to study the behavior of our integral, when integration is conducted according to range $\{\delta < z < 1-\delta\}$. For this purpose we slightly convert integrand in (3.5). We have

$$\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}e^{iT\lambda z}=\frac{1}{4}e^{iT\lambda z}\int_0^T e^{i\tau\lambda}\,d\tau\int_0^T e^{-i\tau\lambda}\,d\tau=$$
$$=\frac{1}{4}\int_0^T e^{i\tau\lambda}\,d\tau\int_{-Tz}^{T(1-z)} e^{-i\tau\lambda}\,d\tau=$$
$$=\frac{1}{4}\left[\int_0^T e^{i\lambda\tau}\,d\tau\int_0^{Tz} e^{i\lambda\tau}\,d\tau+\left|\int_0^{T(1-z)} e^{i\tau\lambda}\,d\tau\right|^2+\right.$$
$$\left.+\int_{T(1-z)}^{T} e^{i\tau\lambda}\,d\tau\int_0^{T(1-z)} e^{-i\tau\lambda}\,d\tau\right] \qquad (3.6)$$

(recall that now $\delta < z, < 1 - \delta$). It is analogous

$$\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}e^{-iT\lambda z}=\frac{1}{4}\left[\int_0^T e^{-i\tau\lambda}\,d\tau\int_0^{Tz} e^{-i\tau\lambda}\,d\tau+\right.$$
$$\left.+\left|\int_0^{T(1-z)} e^{i\tau\lambda}\,d\tau\right|^2+\int_{T(1-z)}^{T} e^{-i\tau\lambda}\,d\tau\int_0^{T(1-z)} e^{i\tau\lambda}\,d\tau\right]. \qquad (3.7)$$

By subtracting from (3.6) equality (3.7), we will obtain finally, that

$$\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}\sin T\lambda z=[\Phi_1(\lambda;\,z)-\Phi_1(-\lambda;\,z)]+$$
$$+[\Phi_2(\lambda;\,z)+\Phi_2(-\lambda;\,z)],$$

where

$$\Phi_1(\lambda;\ z) = \frac{1}{4}\int_0^T e^{i\lambda\tau}\,d\tau \int_0^{Tz} e^{i\lambda\tau}\,d\tau,$$

$$\Phi_2(\lambda;\ z) = \frac{1}{4}\int_{T(1-z)}^{T} e^{i\lambda\tau}\,d\tau \int_0^{T(1-z)} e^{-i\lambda\tau}\,d\tau.$$

end section.

[redacted]

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Let us pass to the estimation of integrals $\frac{1}{T}\int_{-\infty}^{\infty}\Phi_i(\pm\lambda;\ z)$ xf $(\lambda + \mu)\ d\lambda$. The corresponding lining/calculations here even are somewhat simpler than in the discrete case.

1. The estimation $\frac{1}{T}\int_{-\infty}^{\infty}\Phi_1(\lambda;\ z)\,f(\lambda+\mu)\,d\lambda.$

let us register $\Phi_1(\lambda;\ z)$ in the form of sum $\Phi_{11}(\lambda;\ z)+\Phi_{12}(\lambda;\ z)$, where

$$\Phi_{11}(\lambda;\ z)=\frac{1}{4}\int_{T(1-z)}^{T(1-z)+\sqrt{T}} e^{i\lambda\tau}\,d\tau \int_{0}^{T(1-z)} e^{i\lambda\tau}\,d\tau,$$

$$\Phi_{12}(\lambda;\ z)=\frac{1}{4}\int_{T(1-z)+\sqrt{T}}^{T} e^{i\lambda\tau}\,d\tau \int_{0}^{T(1-z)} e^{i\lambda\tau}\,d\tau.$$

It is obvious,

$$|\Phi_{11}(\lambda;\ z)|=\left|\frac{\sin\frac{\lambda\sqrt{T}}{2}}{\lambda}\ \frac{\sin\frac{T(1-z)\lambda}{2}}{\lambda}\right|. \qquad (3.8)$$

Function $\Phi_{12}(\lambda; z)$ can be rewritten in the form

$$\Phi_{12}(\lambda; z) = e^{i\lambda\sqrt{T}}\varphi_1^+(\lambda)\varphi_2^+(\lambda), \qquad (3.9)$$

where

$$|\varphi_1^+(\lambda)| = \left|\frac{\sin(Tz-\sqrt{T})\frac{\lambda}{2}}{\lambda}\right|,$$

$$|\varphi_2^+(\lambda)| = \left|\frac{\sin T(1-z)\frac{\lambda}{2}}{\lambda}\right| \text{ and } \varphi_1^+, \varphi_2^+ \in \mathscr{H}_2^+.$$

We have, further, using property (2.12) of functions h (T), equality (3.8) and a Schwarz inequality:

$$\frac{1}{T}\int_{-\infty}^{\infty}|\Phi_{11}(\lambda; z)|f(\lambda+\mu)\,d\lambda \leqslant$$

$$\leqslant T^{-1/4}[h(\sqrt{T})]^{1/2}[h(T)]^{1/2} = o(h(T)). \qquad (3.10)$$

whereupon this estimation it does not depend on z ($\delta < z < 1 - \delta$).

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The condition of full/total/complete regularity gives taking into account representation (3.9):

$$\left| \int_{-\infty}^{\infty} \Phi_{12}(\lambda;\ z) f(\lambda+\mu)\, d\lambda \right| \leqslant \leqslant \rho(\sqrt{T}) \left[h(Tz-\sqrt{T})\, h(T(1-z)) \right]^{1/2}. \quad (3.11)$$

On lemma 7

$$\lim_{T\to\infty} \frac{h(Tz-\sqrt{T})}{h(T)} = 1, \quad \lim_{T\to\infty} \frac{h(T(1-z))}{h(T)} = 1$$

it is evenly on z, $\delta < z < 1 - \delta$, so that is final evenly on μ and z, $\delta \leqslant z \leqslant 1-\delta$,

$$\left| \frac{1}{T} \int_{-\infty}^{\infty} \Phi_{12}(\lambda;\ z) f(\lambda+\mu)\, d\lambda \right| = o(h(T)). \quad (3.12)$$

Estimations (3.10), (3.12) make it possible to claim that is evenly on $|\mu| \leqslant a$, z, $\delta \leqslant z \leqslant 1-\delta$,

$$\frac{1}{T} \int_{-\infty}^{\infty} \Phi_{1}(\lambda;\ z) f(\lambda+\mu)\, d\lambda = o(h(T)). \quad (3.13)$$

2. Estimation $\frac{1}{T} \int_{-\infty}^{\infty} \Phi_{2}(\lambda;\ z) f(\lambda+\mu)\, d\lambda$.

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Again let us present Φ_2 in the form of the sum

$$\Phi_2(\lambda;\ z) = \Phi_{21}(\lambda;\ z) + \Phi_{22}(\lambda;\ z),$$

where now

$$\Phi_{21}(\lambda;\ z) = \frac{1}{4}\int\limits_{T(1-z)}^{T(1-z)+\sqrt{T}} e^{i\lambda\tau}\,d\tau \int\limits_{0}^{T(1-z)} e^{-i\lambda\tau}\,d\tau,$$

$$\Phi_{22}(\lambda;\ z) = \int\limits_{T(1-z)+\sqrt{T}}^{T} e^{i\lambda\tau}\,d\tau \int\limits_{0}^{T(1-z)} e^{-i\lambda\tau}\,d\tau = e^{i\lambda\sqrt{T}}\varphi_2^+(\lambda)\,\psi_2^+(\lambda), \qquad (3.14)$$

a $\varphi_2^+,\ \psi_2^+ \in \mathscr{H}^2$ of the upper half-plane. From (3.14) we will obtain analogously to [redacted]h(3.10) and (3.12):

$$\frac{1}{T}\int\limits_{-\infty}^{\infty} |\Phi_{21}(\lambda;\ z)|\,f(\lambda+\mu)\,d\lambda \leqslant$$
$$\leqslant T^{-1/4}\left[h(\sqrt{T})\,h(T(1-z))\right]^{1/2} = o(h(T)), \qquad (3.15)$$
$$\frac{1}{T}\left|\int\limits_{-\infty}^{\infty} \Phi_{22}(\lambda;\ z)\,f(\lambda+\mu)\,d\lambda\right| \leqslant$$
$$\leqslant \rho(\sqrt{T})\left[h(Tz-\sqrt{T})\,h(T(1-z))\right]^{1/2} = o(h(T)), \qquad (3.16)$$

so that evenly on $|\mu| \leqslant a$, $\delta \leqslant z \leqslant 1-\delta$

$$\frac{1}{T}\int_{-\infty}^{\infty}\Phi_2(\lambda;\ z)f(\lambda+\mu)\,d\lambda = o(h(T)). \qquad (3.17)$$

Is analogous evenly on μ and z

$$\begin{gathered}\frac{1}{T}\int_{-\infty}^{\infty}\Phi_1(-\lambda;\ z)f(\lambda+\mu)\,d\lambda = o(h(T)),\\ \frac{1}{T}\int_{-\infty}^{\infty}\Phi_2(-\lambda;\ z)f(\lambda+\mu)\,d\lambda = o(h(T)).\end{gathered} \qquad (3.18)$$

These estimations are proven just as estimation (3.13), (3.17), the

only expansion (3.9), (3.18) is substituted the expansions of form $e^{-i\lambda\sqrt{T}}\varphi^{-}(\lambda)\psi^{-}(\lambda)$, where now $\varphi^{-}, \psi^{-} \in \mathcal{H}^2$ in lower half-plane.

By gathering the together obtained estimations, let us find finally that is really/actually evenly on $|\mu| \leqslant a$

$$\frac{1}{T}\int_{-\infty}^{\infty}\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}f(\lambda+\mu)\,a(T\lambda)\,d\lambda = o(h(T)).$$

Lemma is demonstrated.

Lemma 12. Let a (λ) - the turning into zero outside interval [-1, 1] function of bounded variation.

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Then with T $\to \infty$ it is evenly on $|\mu| \leqslant a$

$$\frac{1}{T}\int_{-\infty}^{\infty}\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}f(\lambda+\mu)\,a(T\lambda)\,d\lambda = \\ = \frac{h(T)}{2\pi}\int_{-1}^{1}\frac{\sin^2\frac{\lambda}{2}}{\left(\frac{\lambda}{2}\right)^2}a(\lambda)\,d\lambda\,(1+o(1)). \qquad (3.19)$$

Proof completely coincides with the proof of lemma 14 chapters V.

Let f (λ) - the spectral density of completely regular process. By using lemma 6, let us register f (λ) in the form $w_a(\lambda)|P_a(\lambda)|^2$, where

$$\int_{-\infty}^{\infty} w_a(\lambda)\,d\lambda < \infty, \quad \text{a} \quad \lim_{T} \inf_{|\mu| \leqslant a} \gamma(T;\, \mu;\, w_a) = \infty.$$

On lemma 9 $w_a(\lambda)$ is the spectral density of completely regular process. Further, by using lemma 12, we will obtain that is evenly on $|\mu| \leqslant a$ (comp. page 244)

$$\frac{1}{T}\int_{0}^{1/T} w_a(\lambda+\mu)\,d\lambda - \frac{1}{T}\int_{-1/T}^{0} w_a(\lambda+\mu)\,d\lambda = \\ = o\left(\frac{1}{T}\int_{-1/T}^{1/T} w_a(\lambda+\mu)\,d\lambda\right), \quad (3.20)$$

that equivalent (3.1). The proof of theorem 2 is finished.

§4. Behavior of spectral density on entire straight line.

The demonstrated in the previous paragraph theorem gives no representation of the behavior of spectral density f (λ) of completely regular process with $\lambda \to \infty$.

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It is clear that f (λ), for example, cannot decrease too rapidly.

Since completely regular process is linearly regular,

$$\int_{-\infty}^{\infty} \frac{|\ln f(\lambda)|}{1+\lambda^2} d\lambda < \infty. \qquad (4.1)$$

On the other hand, this, actually, be the only limitation, assigned on the speed of decrease f (λ) with $\lambda \to \infty$. Actually, if $\Gamma(\lambda)$ is the integrated with square integral function of the final degree, then process ξ (t) with spectral density $f(\lambda) = |\Gamma(\lambda)|^2$ is completely regular (this follows from theorem 1).

However, if factor $\Gamma(\lambda)$ above form indicated it is possible to isolate from f (λ) then so that residue/remainder $f(\lambda)(\Gamma(\lambda))^{-1}$ = w (λ) would be with large λ limited on top, also, from below ($m \leqslant w \leqslant M$), then it is possible to draw some conclusions about behavior w (λ) at infinity (more precise, about the uniform behavior w (λ) on entire straight line). A precise formulation result will require the introduction of one class of integral functions, which now will be determined.

In the theory of integral functions the important role plays class A - the class of integral functions (final degree $\Gamma(z)$, $z = \lambda + i\mu$, the final degree, zero z_i which they satisfy the inequality

$$\sum_i |\operatorname{Im} 1/z_i| < \infty. \qquad (4.2)$$

We need integral functions whose zeroes satisfy a condition more rigorous than (4.2).

Here we designate by A* a class of integral functions $\Gamma(z),\ z=\lambda+i\mu,$ of finite power, the zeroes z_l of which satisfy the inequality

$$\sup_{-\infty<\lambda<\infty}\sum_l\left|\operatorname{Im}\frac{1}{z_l-\lambda}\right|<\infty. \qquad (4.3)$$

Here addition common on all the insubstantial zero z_l function Γ.

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Theorem 3. Let spectral density f (λ) the stationary completely regular process ξ (t) allow/assume representation of the form

$$f(\lambda) = |\Gamma(\lambda)|^2 w(\lambda),$$

where Γ - the summarized with square integral function of class A *, and function w satisfies the inequalities

$$0 < m \leqslant \inf_{\lambda} w(\lambda) \leqslant \sup_{\lambda} w(\lambda) \leqslant M < \infty.$$

Then for original W (λ) of function w (λ) is satisfied the condition

$$\omega_W(\delta) = \sup_{-\infty<\lambda<\infty} \sup_{|t|\leqslant\delta} \frac{|W(\lambda+t) + W(\lambda-t) - 2W(\lambda)|}{|W(\lambda+t) - W(\lambda-t)|} \xrightarrow[\delta\to 0]{} 0. \quad (4.4)$$

Proof is realized according to the following plan/layout. First it is proved, that the division during the function of class A * leaves completely regular function by completely regular (relative to terminology see §1). Then completely regular function w, for which, obviously, infγ (T, μ; w) $\xrightarrow[T\to\infty]{}\infty$, it is traced by methods §3.

Let us note that without loss of generality it is possible to consider all the zero functions Γ lying at the locked lower half-plane. Actually, $|\Gamma|^2$ represents on real the direct meaning of the nonnegative integral function of class A, and according to the

theorem of Akhiyezera [1] any this function can be registered in the form $|\phi(\lambda)|^2$, where $\phi(z)$ is the integral function of class A with roots in lower half-plane.

FOOTNOTE [1]. See [16], page 567. ENDFOOTNOTE.

It is understandable that $\varphi \in \Lambda^*$, and if necessary it is possible to replace Γ by ϕ, without varying in this case spectral density f. It is obvious also that function $\Gamma(z)$ can be considered external in the upper half-plane.

Let us assume, further $\Gamma(z)=\overline{\Gamma(\bar{z})}$ let us introduce meromorpmic function $\chi(z)=\frac{\overline{\Gamma}(z)}{\Gamma(z)}$.

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Let us register function $\Gamma(z)$ in the form of the infinite product:

$$\Gamma(z)=z^n e^{az+b}\prod_l\left(1-\frac{z}{z_l}\right)e^{z/z_l}.$$

From condition (4.3) it follows that

$$\chi(z)=\alpha e^{i\beta z}\prod_l\left(1-\frac{z}{\bar{z}_l}\right)\left(1-\frac{z}{z_l}\right)^{-1}. \quad (4.5)$$

where β is real number, a $|\alpha|$ = 1.

Lemma 13. Let integral function $\Gamma(z) \in A^*$, a $\frac{\ln|\Gamma|}{1+\lambda^2} \in \mathscr{L}^1$. $\wedge$ Then meromorpmic function $\chi(z)$ is analytical in certain band $|\mathrm{Im}\, z| < \delta,\ \delta > 0$ and is limited in any band $|\mathrm{Im}\, z| < \delta' < \delta$.

Proof. Without loss of generality it is possible to consider α = 1, β = 0. Let $z_l = \alpha_l + i\beta_l$ on the strength of (4.3)

$$\sup_\lambda \sum \frac{|\beta_l|}{(\lambda-\alpha_l)^2+\beta_l^2} = \sup_\lambda \sum \left| \mathrm{Im}\, \frac{1}{\lambda - z_l} \right| < \infty,$$

and, which means, $\delta = \inf|\beta_l| > 0$, where inf is taken on all insubstantial zeros z_l (if everything zero $\Gamma(z)$ are real, will assume regarding δ = ∞).

It is obvious, function $\chi(z)$ is analytical in band $|\mathrm{Im}\, z| < \delta$. Let δ' < δ; let us demonstrate that $|\chi(z)|$ is limited in $|\mathrm{Im}\, z| < \delta'$. In the band indicated on the strength of (4.3)

$$|\chi(z)|^2 \leqslant \prod_l \left(1 + \frac{4|\beta_l \mu|}{(\alpha_l-\lambda)^2+(\beta_l-\mu)^2}\right) \leqslant$$
$$\leqslant \exp\left\{4|\mu| \sum_l \frac{|\beta_l|}{(\alpha_l-\lambda)^2+(\beta_l-\mu)^2}\right\} \leqslant$$
$$\leqslant \exp\left\{4\delta\left(1-\frac{\delta'}{\delta}\right)^2 \sup_\lambda \sum_l \frac{|\beta_l|}{(\alpha_l-\lambda)^2+\beta_l^2}\right\} = M_{\delta'} < \infty.$$

Lemma 14 under conditions of theorem 3 function w (λ) is

completely regular.

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Proof. From lemma 13 on the basis of the theorems about the approach/approximation of the limited analytic functions [1] escape/ensues the existence of integral functions $\Phi_\sigma(\lambda)$ the final degree $\leqslant\sigma$, for which

$$\|\chi-\Phi_\sigma\|^{(\infty)}=O(e^{-\sigma\delta'}),\qquad \delta'<\delta.$$

FOOTNOTE [1]. See [25], page 317. ENDFOOTNOTE.

With large σ all the all the functions Φ_σ are evenly limited, for example, $|\Phi_\sigma(\lambda)|\leqslant 2$. It is not difficult to comprehend that $e^{i\sigma\lambda}\Phi_\sigma\in\mathscr{H}^\infty$ in the upper half-plane.

Let now ϕ, ψ be arbitrary functions from the single sphere of space $L^+(w)$, then $\varphi,\ \psi\in\mathscr{H}^2$. Consequently, functions $\varphi_1=\varphi/\Gamma,\quad \psi_1=\psi/\Gamma$ will belong to single sphere $L^+(f)$. Finally, $e^{i\sigma\lambda}\Phi_\sigma\varphi_1\in L^+(f)$. Therefore for σ = $\tau/2$

$$\left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\varphi(\lambda)\,\psi(\lambda)\,w(\lambda)\,d\lambda\right| = \left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\varphi_1(\lambda)\,\psi_1(\lambda)\,\chi(\lambda)\,f(\lambda)\,d\lambda\right| \leqslant$$
$$\leqslant \left|\int_{-\infty}^{\infty} e^{i\lambda\tau/2}\left[e^{i\lambda\tau/2}\Phi_\sigma(\lambda)\,\varphi_1(\lambda)\right]\psi_1(\lambda)\,f(\lambda)\,d\lambda\right| + \int_{-\infty}^{\infty} |\varphi_1(\lambda)\,\psi_1(\lambda)|\,|\chi(\lambda)-\Phi_\sigma(\lambda)|\,f(\lambda)\,d\lambda \leqslant$$
$$\leqslant 2\rho(\tau/2) + \|\chi-\Phi_\sigma\|^{(\infty)}.$$

Hence in turn, it follows that

$$\rho(\tau,\ w) \leqslant 2\rho(\tau/2) + O(e^{-\tau\delta'/2}), \qquad \delta' < \delta.$$

Lemma is demonstrated.

The remaining part of the proof of theorem 3 in accuracy coincides with the proof of theorem 2.

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Actually, according to the condition of the theorem

$$\inf \gamma(T;\ x;\ w) \geqslant \frac{m\pi}{2} T \xrightarrow[T \to \infty]{} \infty.$$

Furthermore, the function of form $\int_0^a e^{it\lambda}\,dt$ essence cell/elements from $L^+(w)$. Consequently, the proofs of all lemmas 7, 8, 10-12 remain valid. In addition now in all these lemmas it is possible to place a = ∞.

Condition (4.4) indicates in general terms the limitedness of relation $\left|\frac{w'(\lambda)}{w(\lambda)}\right|$ or, more general, uniform continuity on the entire axis ln w (λ) (comp. the following paragraph). This condition can be broken even for very smooth functions. Let us consider the following.

Example. Let the process ξ (t) have as its spectral density the function

$$f(\lambda) = (\sin^2 \lambda^2 + 1)\left(\frac{\sin \lambda}{\lambda}\right)^{2p},$$

where p - arbitrary positive integer number. It is not difficult to see that $\int_{-\infty}^{\infty} \frac{\ln f(\lambda)}{1+\lambda^2}\,d\lambda < \infty$, so that process ξ (t) is regular. However, it is not completely regular, although its spectral density is analytical in an entire plane of complex variable $z = \lambda + i\mu$ and has zeros coinciding on real direct/straight from zero integral function final degree $\Gamma(z) = \left(\frac{\sin z}{z}\right)^{2p}$.

Actually, function $\Gamma(z) \in A^*$: it has only real zeros. In accordance with theorem 3 function w (λ) = sin²λ² + 1 must satisfy condition

(4.4). However,

$$\frac{|W(\lambda+t)+W(\lambda-t)-2W(\lambda)|}{W(\lambda+t)-W(\lambda-t)} \geqslant$$
$$\geqslant \frac{1}{2t}\left|\int\limits_{\lambda}^{\lambda+t} [\sin^2 s^2 - \sin^2 (s-t)^2]\, ds\right| =$$
$$= \frac{1}{2t}\left|\int\limits_{\lambda}^{\lambda+t} \sin t(2s-t) \sin [s^2+(s-t)^2]\, ds\right|. \quad (4.6)$$

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Let us select in (4.6) $\lambda = \pi/4t$. Then for $\lambda \leqslant s \leqslant \lambda + t$ with $t \to 0$

$$\sin t(2s-t) = 1 + O(t^2),$$
$$\sin (s^2+(s-t)^2) = -\cos 2s^2 + O(t^2).$$

Consequently,

$$\frac{|W(\lambda+t)+W(\lambda-t)-2W(\lambda)|}{W(\lambda+t)-W(\lambda-t)} \geqslant \frac{1}{2t}\left|\int\limits_{\lambda}^{\lambda+t} \cos 2s^2\, ds\right| + O(t^2).$$

Further,

$$\int\limits_{\lambda}^{\lambda+t} \cos(2s^2)\, ds = \int\limits_{\lambda}^{\lambda+t} \frac{d \sin(2s^2)}{s} =$$
$$= \frac{\sin 2(\lambda+t)^2}{\lambda+t} - \frac{\sin 2\lambda^2}{\lambda} + \int\limits_{\lambda}^{\lambda+t} \frac{\sin 2s^2}{s^2}\, ds =$$
$$= -\frac{2}{\lambda} \sin \frac{\pi^2}{8t^2} + O(|t|^3).$$

Thus, finally we have

$$\sup_{\lambda} \frac{|W(\lambda+t)+W(\lambda-t)-2W(\lambda)|}{W(\lambda+t)-W(\lambda-t)} \geqslant$$
$$\geqslant \frac{4}{\pi}\left|\sin\frac{\pi^2}{8t^2}\right| + O(t^2) \nrightarrow 0.$$

Thereby it is establish/installed that the process ξ (t) is not completely regular.

§5. Sufficient conditions.

One of the criteria of full/total/complete regularity is given by the following theorem, which is partly converse theorem 3.

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Theorem 4. Let spectral density f (λ) of stationary process $\xi(t)$ allow/assume representation of the form

$$f(\lambda) = |\Gamma(\lambda)|^2 w(\lambda),$$

where Γ - the summarized with square integral function of the final degree $\leqslant \sigma$, a function w possesses the following properties:

1) $0 < m \leqslant w(\lambda) \leqslant M < \infty$;

2) $\sum_n \omega_W^2(2^{-n}) < \infty$; here, as above,

$$\omega_W(\delta) = \sup_{\lambda} \sup_{|t| \leqslant \delta} \frac{|W(\lambda + t) + W(\lambda - t) - 2W(\lambda)|}{|W(\lambda + t) - W(\lambda - t)|},$$

a W is original for w.

Then process ξ (t) is completely regular, whereupon

$$\rho(\tau) \leqslant C\left(\frac{M}{m}\right)^3 \left(\sum_1^{\infty} \omega_W^2\left(\frac{1}{\tau - 2\sigma} 2^{-(n-1)}\right)\right)^{1/2},$$

where C is absolute constant.

We will drop/omit proof, because it only in terms of technicality differs from the proof of the corresponding discrete analog - theorem 6 chapters V.

Theorem 5. Let spectral density f (λ) of process ξ (t) allow/assume representation of the form f (λ) $= |\Gamma(\lambda)|^2 w(\lambda)$, where 1) Γ - the limited integral function of the final degree ≤σ;

2) function lnw (λ) is evenly continuous on (-∞, ∞), i.e., $\sup_{\lambda,\, |h| \leqslant s} \left| \ln \frac{w(\lambda+h)}{w(\lambda)} \right| = \omega(s) \xrightarrow[s \to 0]{} 0;$

3) $\int\limits_{1}^{\infty} \frac{\omega(s)}{s^2}\, ds < \infty.$

Then process ξ (t) is completely regular, whereupon

$$\rho(\tau) \leqslant C\omega\left(\frac{1}{\tau - 2\sigma}\right), \qquad \tau > 2\sigma \qquad (5.1)$$

(where constant C depends on w).

On the strength of lemma 1 it suffices to demonstrate full/total/complete regularity of function w (λ). From inequality $|\ln w(\lambda)| \leqslant |\ln w(0)| + \omega(|\lambda|)$ and condition 3) theorem it follows that $\frac{\ln w}{1+\lambda^2} \in \mathscr{L}^1$. It remains to consider value ρ (r, w). This estimation is based on the following basic lemma.

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To lemma 15. Under conditions of theorem 5 for any r > 0 will be

located nonnegative function $\Phi_r(\lambda)$ such that

1) for all $\varphi,\ \psi \in L^+(w)$ the function

$$\varphi\Phi_r^{1/2},\ \psi\Phi_r^{1/2} \in L^2(-\infty,\ \infty),$$

and if, furthermore, $\tau > r$, then,

$$\int_{-\infty}^{\infty} \varphi(\lambda)\psi(\lambda)e^{i\lambda\tau}\Phi_r(\lambda)\,d\lambda = 0; \qquad (5.2)$$

2) for all $\lambda, -\infty < \lambda < \infty$,

$$|w(\lambda) - \Phi_r(\lambda)| \leqslant C_1\omega(1/r)\,w(\lambda). \qquad (5.3)$$

After plotting thus far the proof of lemma, let us show, as from it is derive/concluded inequality (5.1). Let $\phi(\lambda)$, $\psi(\lambda)$ - be arbitrary functions from the single sphere of space $L^+(w)$. On the basis of lemma for all $\tau > r$

$$\left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\varphi(\lambda)\psi(\lambda)w(\lambda)\,d\lambda\right| =$$
$$= \left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\varphi(\lambda)\psi(\lambda)[w(\lambda) - \Phi_r(\lambda)]\,d\lambda\right| \leqslant$$
$$\leqslant C_1\omega(1/r)\int_{-\infty}^{\infty} |\varphi(\lambda)||\psi(\lambda)|w(\lambda)\,d\lambda \leqslant C_1\omega(1/r).$$

Consequently, $\rho(r) \leqslant C_1\omega(1/r)$, and reference to lemma 1 proves inequality (5.1).

Let us return to lemma 15. Let us determine number α by the equality

$$\alpha = \sum_{n=1}^{\infty} \frac{\omega(n)}{n^2}.$$

In view of condition 3) theorem 5 determining α series descends.

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Let us assume, further $a_k = \frac{\omega(n)}{2\alpha n^2}$,

$$k(x) = \frac{\sin^4 \frac{x}{8}}{\left(\frac{x}{8}\right)^r} \prod_1^{\infty} \frac{\sin^2 \frac{a_k x}{2}}{\left(\frac{a_k x}{2}\right)^2}$$

let us determine "nucleus Marchenko" [1] K (x), by set/assuming

$$K(x) = k(x) \left(\int_{-\infty}^{\infty} k(x)\, dx \right)^{-1}.$$

FOOTNOTE [1]. These nuclei are introduced to V. A. Marchenko in the work "on some questions of the approximation of the continuous

functions on all of real axis", III, the report/communications Kharkov Mathematics Society XXII (1950). ENDFOOTNOTE.

The sought functions $\Phi_r(\lambda)$ now are determined by the equality

$$\Phi_r(\lambda) = r\int_{-\infty}^{\infty} K(rx)\, w(\lambda - x)\, dx.$$

In order to demonstrate that Φ_r they satisfy all requirements for lemma 15, is preliminarily studied the property of nuclei K (x).

Lemma 16. Nuclei K (x) possess the following properties:

1) K (x) - the integrated with square integral function of the final degree $(\leqslant {}^1/_2 + \sum a_k \leqslant 1)$;

2) $\int_{-\infty}^{\infty} K(x)\, dx = 1,$

3) with any $r > 0$

$$|K(rx)| \leqslant M_r e^{-\omega(x)} \frac{\sin^4 \frac{rx}{8}}{\left(\frac{rx}{8}\right)^4}, \tag{5.4}$$

where the constant M_r it depends only on r.

two first properties are obvious. In order to demonstrate the latter, let us note first that $\lim\limits_{x\to\infty}\frac{\omega(x)}{x}=0.$ Actually, from the convergence of integral $\int\limits_1^\infty \frac{\omega(x)}{x^2}\,dx$ it follows that $\underline{\lim\limits_{x\to\infty}}\frac{\omega(x)}{x}<\infty.$

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By integrating in parts, let us arrive at the equality

$$\int\limits_1^x \frac{\omega(y)}{y^2}\,dy=\int\limits_1^x \frac{d\omega(y)}{y}-\frac{\omega(x)}{x}+\omega(1),$$

from which it follows that $\underline{\lim\limits_{x}}\int\limits_1^x \frac{d\omega(y)}{y}<\infty.$ Since the function $\omega(x)$ does not decrease, the last/latter condition will draw the convergence of integral $\int\limits_1^\infty \frac{d\omega(y)}{y}.$ Consequently, there exists $\lim\frac{\omega(x)}{x}$, which on the strength of the convergence of integral $\int\limits_1^\infty \frac{\omega(x)}{x^2}\,dx$ is equal to zero.

Without loss of generality it is possible to count $\lim\limits_{x\to\infty}\omega(x)=\infty$ (otherwise inequality (5.4) trivially). On assigned $x > 0$, let us select integer N so, in order to $N \leqslant \omega(x) < N + 1$. Then

$$\prod_1^\infty \frac{\sin^2\frac{a_k xr}{2}}{\left(\frac{a_k xr}{2}\right)^2}\leqslant\prod_1^N\left(\frac{2}{a_k x}\right)^2=\frac{2^{2N}(N!)^4}{x^{2N}\prod\limits_1^N \omega^2(k)}\,\frac{(2a)^{2N}}{r^{2N}}.$$

On the basis of the Stirling formula with large x

$$\frac{(N!)^4}{x^{2N}}\leqslant N^{4N}x^{-2N}\leqslant e^{4\omega(x)\ln\frac{1}{\omega(x)}}\,e^{-2(\omega(x)-1)\ln x}.$$

The last/latter expression with large x will not exceed $e^{-\omega(x)}$ (since $\frac{\omega(x)}{x} \to 0$).

Furthermore, on the strength of equality $\lim\limits_{k\to\infty} \omega(k) = \infty$, obviously and

$$\lim \frac{(2a)^{2N} 2^{2N} r^{-2N}}{\omega^2(1) \ldots \omega^2(N)} = 0.$$

Inequality (5.4) is proved.

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Being returned to functions Φ_r, let us note that on the strength of p. 3) lemmas 16

$$\Phi_r(\lambda) \leqslant r w(\lambda) \int_{-\infty}^{\infty} K(rx) \sup_{\lambda} \frac{w(\lambda - x)}{w(\lambda)} dx \leqslant$$
$$\leqslant r w(\lambda) \int_{-\infty}^{\infty} K(rx) e^{\omega(|x|)} dx \leqslant C_2 M_r w(\lambda), \quad (5.5)$$

so that for all φ, for which $\int |\varphi(\lambda)| w(\lambda) d\lambda < \infty$, is compulsory $\varphi\Phi_r \in \mathscr{L}^1$. Let us assume, further

$$\Phi_{rT}(\lambda) = r \int_{-T}^{T} K(r(x - \lambda)) w(x) dx, \quad T > 0.$$

It is obvious, with all T

$$\Phi_{rT}(\lambda) \leqslant \Phi_r(\lambda) \leqslant \pi M_1 w(\lambda). \quad (5.6)$$

If, further, $|\lambda| \leqslant T/2$, then it is analogous to [redacted](5.5)

$$\Phi_r(\lambda) - \Phi_{rT}(\lambda) =$$
$$= r\left(\int\limits_{-\infty}^{-T-\lambda} K(rx)\,w(x-\lambda)\,dx + \int\limits_{T-\lambda}^{\infty} K(rx)\,w(x-\lambda)\,dx\right) \leqslant$$
$$\leqslant rw(\lambda)\left[\int\limits_{-\infty}^{-T/2} K(rx)\,e^{\omega(x)}dx + \int\limits_{T/2}^{\infty} K(rx)e^{\omega(x)}dx\right] \leqslant$$
$$\leqslant \frac{C_3 w(\lambda)}{T} M_r.$$

Hence and from (5.6) it follows that for any fixed/recorded functions $\varphi,\ \psi \in L(w)$ and any fixed/recorded r

$$\lim_{T\to\infty} \int\limits_{-\infty}^{\infty} \varphi(\lambda)\,\psi(\lambda)\,e^{i\lambda\tau}\Phi_{rT}(\lambda)\,d\lambda =$$
$$= \int\limits_{-\infty}^{\infty} \varphi(\lambda)\,\psi(\lambda)\,e^{i\lambda\tau}\Phi_r(\lambda)\,d\lambda.$$

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The last/latter equality together with the determination of space $L^+(w)$ allows in the proof of equality (5.2) to be bounded to the study of the integrals of the form

$$\int\limits_{-\infty}^{\infty} \varphi(\lambda)\,\psi(\lambda)\,e^{i\lambda\tau}\Phi_{rT}(\lambda)\,d\lambda, \qquad (5.7)$$

where $\varphi, \psi \in \mathscr{H}^2 \cap L(w)$, $\tau > r$, and T is any positive number.

According to the theorem of Peli - Weiner, the Fourier transform $\chi_r(x)$ nucleus rK (rx) is converted into zero outside interval [-r, r], so that

$$\Phi_{rT}(\lambda) = \int_{-T}^{T} w(x)\,dx \int_{-r}^{r} e^{i(\lambda - x)s} \chi_r(s)\,ds =$$
$$= \int_{-r}^{r} e^{i\lambda s} b(s)\,ds, \quad b \in \mathscr{L}^2.$$

Consequently, according to the same theorem of Peli - Weiner whole $\Phi_{rT}(\lambda)$ essence the summarized with square integral functions of the final degree $\leqslant$r. Therefore with all τ > r of day lily $e^{i\lambda\tau}\varphi\psi\Phi_{rT} \in \mathscr{H}^1$ in the upper half-plane and all integrals (5.7) are equal to zero. Equality (5.2) is proved.

For the proof of the second part of lemma 15 - inequality (5.3) - let us note first that on the analogy 2) nuclei K (x)

$$|w(\lambda) - \Phi_r(\lambda)| \leqslant \int_{-\infty}^{\infty} K(x) \left| w(\lambda) - w\left(\lambda - \frac{x}{r}\right) \right| dx \leqslant$$
$$\leqslant w(\lambda) \int_{-\infty}^{\infty} K(x) \sup_{\lambda} \left| \frac{w(\lambda) - w\left(\lambda - \frac{x}{r}\right)}{w(\lambda)} \right| dx. \quad (5.8)$$

With all x $|e^x - 1| \leqslant |x| e^{|x|}$. Therefore

$$\left| \frac{w(\lambda) - w\left(\lambda - \frac{x}{r}\right)}{w(\lambda)} \right| = \left| 1 - \exp\left\{ \ln \frac{w\left(\lambda - \frac{x}{r}\right)}{w(\lambda)} \right\} \right| \leqslant \leqslant \omega\left(\frac{|x|}{r}\right) e^{\omega\left(\frac{|x|}{r}\right)}.$$

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Let us substitute the obtained inequality in (5.8). Since $\omega(|x|/r) \leqslant (1 + |x|)\,\omega(1/r)$, we will have (taking into account the property (5.4) of nuclei K (x)), that for all r > 1

$$|w(\lambda) - \Phi_r(\lambda)| \leqslant w(\lambda)\, w(1/r) \int_{-\infty}^{\infty} K(x)(1 + |x|)\, e^{\omega(|x|)}\, dx \leqslant \leqslant M_1 \omega(1/r) \int_{-\infty}^{\infty} (1 + |x|) \frac{\sin^4 \frac{x}{8}}{(x/8)^4}\, dx \cdot w(\lambda) = C_1 \omega(1/r)\, w(\lambda).$$

FOOTNOTE [4]. Any module/modulus of continuity $\omega(\delta)$ satisfies inequality $\omega(\gamma\delta) \leqslant (1+\gamma)\,\omega(\delta),\ \gamma > 0$ (for example, see [25], page 213). ENDFOOTNOTE.

Lemma 15, and with it and theorem 5 are demonstrated completely.

Observation. As can be seen from the course of proof, constant C_3 can be registered in the form C_4M_1, where this time C_4 - the absolute constant, and M_1 - the characteristic of nucleus K - is determined by function $\omega(x)$. Therefore, although the constant M_1 and is not absolute, it it is possible to select one and the same for the whole class of functions $w(\lambda)$, for example for all those $w(\lambda)$, for

which

$$\sup_{\lambda}\left|\ln\frac{g(\lambda+x)}{g(\lambda)}\right|\leqslant\omega(x).$$

Example. Let us consider stationary **process** $\xi_a(t)$ **with spectral density** $f_a(\lambda)=e^{-|\lambda|^a},\ a>0.$ **If** $a \geqslant 1$**, then** $\frac{f_a}{1+\lambda^2}\notin\mathscr{L}^1$, **so that process** $\xi_a(t)$ **even is not regular. If** $a < 1$**, then**

$$\sup_{\lambda}\ln\frac{f_\alpha(\lambda+x)}{f_\alpha(\lambda)}=\sup_{\lambda}(|\lambda+x|^\alpha-|\lambda|^\alpha)\leqslant\alpha|x|^\alpha.$$

This follows from the easily checked inequality

$$(1+u)^\alpha-1-\alpha u^\alpha\leqslant 0,\quad u>0.$$

Consequently with $a < 1$ process $\xi_a(t)$ **is completely regular, a** $\rho(\tau)=O(\tau^{-a})$. (It goes without saying that constant in symbol O (•) **depends on** a**).**

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§6. one special class of stationary processes.

In this paragraph will be shown the conditions of the full/total/complete re**gularity of stationary** processes ξ (t) with the spectral density of **form** $|\Gamma(\lambda)|^{-2}$, **where** $\Gamma(\lambda)$ is the integral function of the final degree. How **is explained interest** in this special form of

processes?

most simply tripled, apparently, one should consider the stationary processes, spectral density of which is equal to $|P(\lambda)|^{-2}$, P-polynomial. In the case of Gaussian stationary processes the processes of the form indicated are contained by all projections of Markov vector processes (continuous analogs m-member Markov chains). From theorem 5 immediately it follows that the stationary process with density $|P(\lambda)|^{-2}$ is completely regular. Simple calculations show that the corresponding coefficient of regularity satisfies condition [1]

$$\rho(\tau) = O(e^{-\tau(\delta-\varepsilon)}), \qquad (6.1)$$

where 2δ is width of the band of analyticity $(|\mathrm{Im}\, z| < \delta)$ function 1/P (z), a ε is an arbitrary positive number.

FOOTNOTE [1]. In the article of A. M. Yaglom, cited on page 162, it is proved that in this case $\rho(\tau)$ there is the maximum root of certain determinant of equation; other roots of this equation also coincide with the eigenvalues of operator B_τ. ENDFOOTNOTE.

The integral functions of the final degree are the following in complexity class analytic functions after polynomials, and it is

possible to think which in this situation is still possible to find idle time and final criterion of full/total/complete regularity. If this it were possible do, simultaneously will be solved the appearing naturally problem of the description of that set of the integral functions of the final degree, division into which retains complete regularity (see §§2-4; comp. also with lemma 11 chapters V).

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Theorem 6. Stationary process ξ (t) with the spectral density of form $|\Gamma(\lambda)|^{-2}$, where Γ - the integral function of the final degree, is completely regular in that and only that case, if

1) $\frac{\ln|\Gamma|}{1+\lambda^2} \in \mathscr{L}^1(-\infty, \infty)$;

2) function Γ is a function of class [1] A *, i.e., $\sup_\lambda \sum_\Lambda \left|\operatorname{Im}\frac{1}{\lambda - z_j}\right| < \infty$, where the addition is conducted according to by all insubstantial zero z_j function Γ.

FOOTNOTE [1]. Recall that the class A * is determined in §4. ENDFOOTNOTE.

In this case it is necessary $\inf|\operatorname{Im} z_j| = \delta > 0$ and with all $\varepsilon > 0$

$$\rho(\tau) = O(e^{-\tau(\delta-\varepsilon)}). \qquad (6.2)$$

Proof. Let be satisfied conditions 1) and 2). Without loss of generality it is possible to count that all the zero functions $\Gamma(z)$ lie/rest at lower half-plane, a $\Gamma(z)$ and $\frac{1}{\Gamma(z)}$ essence external functions in the upper half-plane (see §4). Let us introduce meromorpmic function $\chi(z)=\frac{\Gamma(z)}{\bar{\Gamma}(z)}$, where as before $\bar{\Gamma}(z)=\overline{\Gamma(\bar{z})}$.

On the strength of equality (1.4)

$$\rho(\tau)=\sup_{\theta}\left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\theta(\lambda)\chi(\lambda)\,d\lambda\right|, \qquad (6.3)$$

where θ passes the single sphere of space $\mathscr{H}^1$. On lemma 13 function $\chi(z)$ is analytical in the band of nonzero width $|\operatorname{Im} z|<\delta$, $\delta=\inf|\operatorname{Im} z_j|>0$, and is limited in any band $|\operatorname{Im} z|\leqslant\delta'<\delta$. On the basis of S. N. Bernstein's theorems about the approach/approximation analytic functions [2] will be located the limited integral functions $\Phi_r(\lambda)$ the final degree $\leqslant r$, for which

$$\sup_{\lambda}|\chi(\lambda)-\Phi_r(\lambda)|=O(e^{-r(\delta-\varepsilon)}), \quad \varepsilon>0.$$

FOOTNOTE [2]. See [25], page 317. ENDFOOTNOTE.

For all r < τ

$$\int_{-\infty}^{\infty} e^{i\lambda\tau}\theta(\lambda)\,\Phi_r(\lambda)\,d\lambda = 0.$$

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Therefore for r < τ

$$\rho(\tau) = \sup_{\theta}\left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\theta(\lambda)\left[\chi(\lambda) - \Phi_r(\lambda)\right]d\lambda\right| = O\left(e^{-r(\delta-\varepsilon)}\right),$$

that also proves equality (6.2).

The need for condition 1) follows from the regularity of process ξ (t), and again it is possible to count that Γ and $1/\Gamma$ essence external functions in the upper half-plane.

Let us demonstrate the need for condition $\Gamma \in A'$. If we designate by z_i zero functions $\Gamma(z)$, it it will be possible to register in the form

$$\Gamma(z) = e^{az+b}\prod_{j}\left(1 - \frac{z}{z_j}\right)e^{z/z_j}. \qquad (6.4)$$

From already demonstrated condition 1) of the theorem it follows that 1

$$\sum\left|\operatorname{Im}\frac{1}{z_j}\right| < \infty.$$

FOOTNOTE [1]. See [16], page 314. ENDFOOTNOTE.

Therefore

$$\chi(z)=\frac{\Gamma(z)}{\bar{\Gamma}(z)}=\alpha e^{i\beta z}\prod\left(1-\frac{z}{z_j}\right)\left(1-\frac{z}{\bar{z}_j}\right)^{-1},$$

where β is real number, a $|\alpha|=1$. In the upper half-plane $z=\lambda+i\mu$, $\mu>0$,

$$|\bar{\chi}(z)e^{i\beta z}|=\left|\frac{e^{i\beta z}}{\chi(z)}\right|=\prod\left|1-\frac{z}{\bar{z}_j}\right|\left|1-\frac{z}{z_j}\right|^{-1}=$$
$$=\prod\left[\frac{(\operatorname{Re} z_j-\lambda)^2+(\operatorname{Im} z_j+\mu)^2}{(\operatorname{Re} z_j-\lambda)^2+(\operatorname{Im} z_j-\mu)^2}\right]^{1/2}\leqslant 1.$$

Consequently, for all $\tau \geqslant \beta$ function $e^{i\tau z}\bar{\chi}(z)$ is (internal) the function of class $\mathscr{H}^\infty$. Therefore every time that $\theta\in\mathscr{H}^1$, function $e^{i\tau\lambda}\theta\bar{\chi}$ also belongs $\mathscr{H}^1$ in the upper half-plane, so that

$$\int_{-\infty}^{\infty} e^{i\tau\lambda}\theta(\lambda)\bar{\chi}(\lambda)\,d\lambda=0. \qquad (6.5)$$

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At the same time (on 1.4))

$$\sup_{\theta \in H^1} \left| \int_{-\infty}^{\infty} e^{i\lambda\tau} \theta(\lambda) \chi(\lambda) \, d\lambda \right| = \rho(\tau). \qquad (6.6)$$

Let us note that if $\theta \in \mathscr{H}^1$ in the upper half-plane, then $\bar{\theta} \in \mathscr{H}^1$ in lower half-plane. Therefore equalities (6.5) and (6.6) make it possible to claim that for all θ^+, those belonging to single sphere $\mathscr{H}^{1+}$, and θ^-, that belong to single sphere $\mathscr{H}^{1-}$, and all $\tau \geqslant \beta$

$$\begin{aligned} \left| \int_{-\infty}^{\infty} e^{i\lambda\tau} \theta^+(\lambda) \chi(\lambda) \, d\lambda \right| &\leqslant \rho(\tau), \\ \left| \int_{-\infty}^{\infty} e^{-i\lambda\tau} \theta^-(\lambda) \chi(\lambda) \, d\lambda \right| &\leqslant \rho(\tau). \end{aligned} \qquad (6.7)$$

Inequalities (6.7) make it possible to demonstrate the following lemma.

Lemma 17. It is evenly on $\lambda, -\infty < \lambda < \infty$, with $t \to 0$

$$\int_{\lambda-t}^{\lambda} \chi(s) \, ds - \int_{\lambda}^{\lambda+t} \chi(s) \, ds = o(t). \qquad (6.8)$$

Proof. Let a (λ) - the odd, three times differentiated function, which turns into zero outside interval [-1, 1]. Let us demonstrate first that is evenly on x,-∞ < x < ∞, with T → ∞

$$\frac{1}{T}\int_{-\infty}^{\infty}\frac{\sin^2\frac{T\lambda}{2}}{\lambda^2}\chi(\lambda+x)\,a(T\lambda)\,d\lambda=o(1). \qquad (6.9)$$

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This equality is the analog of lemma 11. On the strength of (6.7) they occur of the inequality

$$\left|\int_{-\infty}^{\infty}e^{i\lambda\tau}\left(\int_{0}^{T}e^{i\lambda u}\,du\right)^2\chi(\lambda+x)\,d\lambda\right|\leqslant$$

$$\leqslant\rho(\tau)\int_{-\infty}^{\infty}\frac{\sin^2\frac{\lambda T}{\lambda^2}}{\lambda^2}\,d\lambda=\rho(\tau)\frac{\pi T}{2}, \qquad (6.10)$$

$$\left|\int_{-\infty}^{\infty}e^{-i\lambda\tau}\left(\int_{-T}^{T}e^{i\lambda u}\,du\right)\chi(\lambda+x)\,d\lambda\right|\leqslant\rho(\tau)\frac{\pi T}{2},$$

if only τ > β. Let us note now that precisely inequalities of type

(6.10) are the basis of the proof of lemma 11. It is literal the same arguments that they prove lemma 11, they make it possible to deduce equality (6.9) from inequalities (6.10). Therefore here this conclusion can be drop/omitted.

Further, let us determine the odd function $a_0(\lambda)$ by the equalities:

$$a_0(\lambda) = \begin{cases} \dfrac{\lambda^2}{\sin^2 \dfrac{\lambda}{2}}, & 0 < \lambda < 1, \\ 0, & \lambda > 1, \end{cases}$$
$$a_0(\lambda) = -a_0(-\lambda).$$

Let $a_\varepsilon(\lambda)$, $\varepsilon > 0$ - the odd, three times differentiated functions, which coincide with $a_0(\lambda)$ outside intervals $[-\varepsilon, \varepsilon]$, $[-1, -1+\varepsilon]$, $[1-\varepsilon, \varepsilon]$, whereupon within the indicated intervals ~~of~~ functions $a_\varepsilon(\lambda)$ are monotonic. On the strength of (6.9)

$$\left| \frac{1}{T} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{T\lambda}{2}}{\lambda^2} \chi(\lambda + x) [a_0(T\lambda) - a_\varepsilon(T\lambda)] \, d\lambda \right| \leqslant$$
$$\leqslant \frac{2}{\sin^2 \frac{1}{2}} \int_{-\varepsilon}^{\varepsilon} \frac{\sin^2 \frac{\lambda}{2}}{\lambda^2} d\lambda + o(1) \leqslant 16\varepsilon + o(1).$$

Consequently, equality (6.9) will remain in force, if we there replace a (λ) by a_0 (λ); designating still 1/T through t, we will obtain (6.8). Lemma is demonstrated.

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Lemma 18. There is a positive number δ > 0 such, that the function $\chi(z)=\Gamma(z)/\bar{\Gamma}(z)$ is analytical in band $|\mathrm{Im}\, z|<\delta$.

Proof. Function $\chi(z)$ is meromorpmic, and as its poles serve insubstantial zero functions $\bar{\Gamma}(z)$. Consequently, it is to demonstrate that all the insubstantial zero functions $\bar{\Gamma}(z)$ lie/rest outside band $|\mathrm{Im}\, z|<\delta$. Let $\bar{z}_l=\alpha_l-i\beta_l$ be any zero functions $\bar{\Gamma}$.

Let us determine integral functions $\gamma_l(z)$ and $\bar{\gamma}_l(z)$ by the equalities

$$\bar{\gamma}_l(z)=\left(1-\frac{z}{\bar{z}_l}\right)^{-1}\bar{\Gamma}(z),\quad \gamma_l(z)=\overline{\bar{\gamma}(\bar{z})}.$$

Let, further,

$$\varphi_l(\lambda)=\frac{1}{2\pi T_l}\,\frac{e^{iT_l(\lambda-\alpha_l)}-1}{\lambda-\alpha_l},\quad T_l=\frac{1}{|\beta_l|}.$$

It is not difficult to comprehend that $\bar{\gamma}_l/\gamma_l\in\mathscr{H}^\infty$ in the upper half-plane, and therefore $\varphi_l\frac{\bar{\gamma}_l}{\gamma_l}\in\mathscr{H}^1$ in the upper half-plane, whereupon $\left\|\varphi_l\frac{\bar{\gamma}_l}{\gamma_l}\right\|^{(1)}=1$. On the strength of (6.7)

$$\left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\varphi_l(\lambda)\frac{\bar{\gamma}_l(\lambda)}{\gamma_l(\lambda)}\chi(\lambda)\,d\lambda\right| = \\ = \left|\int_{-\infty}^{\infty} e^{i\lambda\tau}\varphi_l(\lambda)\frac{z_l-\lambda}{\bar{z}_l-\lambda}\,d\lambda\right| \leqslant \rho(\tau). \quad (6.11)$$

Function $\varphi_l(z)\frac{z_l-z}{\bar{z}_l-z}$ is analytical in the upper half-plane, with the exception of pole at point z_l, and it approaches 0 with $|z|\to\infty$. Therefore integral in right side can be computed according to the deductions of integrand. It is equal to

$$2\pi i\left(\overset{(1)}{\text{вычет}}\, e^{i\lambda\tau}\varphi_l(z)\frac{z_l-z}{\bar{z}_l-z}\right) = -2(e^{-1}-1)^2 e^{i\tau\bar{z}_l}.$$

Key: (1). residue.

Substituting this result in (6.11), let us find that

$$\rho(\tau) \geqslant \frac{2}{e^2}(e-1)^2 e^{\beta_l\tau}. \quad (6.12)$$

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Consequently, if $\delta = \inf|\beta_l|$, where inf is taken on all $\beta_l \neq 0$, then on the basis (6.12)

$$\rho(\tau) \geqslant \frac{2}{e^2}(e-1)^2 e^{-\delta\tau},$$

and, then with an increase τ ρ (τ) $\to$ 0, it is necessary δ > to 0. Lemma 18 is demonstrated.

Now we will show that from equality (6.8) and the analyticity of function $\chi(z)$ in band $|\mathrm{Im}\, z|<\delta$ follows the inequality for a derivative $\chi'(\lambda)$: $\sup_\lambda|\chi'(\lambda)|<\infty$. Since

$$|\chi'(\lambda)|=\left|\beta-\sum \mathrm{Im}\,\frac{1}{z_j-\lambda}\right|=\left|\beta-2\sum\left|\mathrm{Im}\,\frac{1}{z_j-\lambda}\right|\right|,$$

the theorem will be demonstrated. Subsequently (for simplicity) β will be assumed to be equal to zero. This does not diminish generality, because in basic inequalities (6.7) factor $e^{\pm i\lambda\beta}$ can be introduced into composition $e^{i\tau\lambda}$, which in the worse case will bring perhaps only to replacement ρ (τ) on $\rho(\tau-|\beta|)$.

Let as be above, $z_j=\alpha_j+i\beta_j$. On the basis of Leibnitz's formula

$$\chi'(\lambda)=\chi(\lambda)\sum\frac{|2\beta_j|}{(\alpha_j-\lambda)^2+\beta_j^2},$$
$$\chi^{(s+1)}(\lambda)=\sum_0^s C_s^k\chi^{(k)}(\lambda)\left(\sum_j\frac{2|\beta_j|}{(\alpha_j-\lambda)^2+\beta_j^2}\right)^{(s-k)}, \qquad (6.13)$$

whereupon the summation is conducted according to that j, for which $\beta_j \neq 0$. On lemma 18 $|\beta_j|=-\beta_j\geqslant\delta>0$, and therefore

$$\left|\left(\frac{2\beta_j}{(\alpha_j-\lambda)^2+\beta_j^2}\right)^{(r)}\right| = \left|\left(\frac{1}{\bar{z}_j-\lambda}-\frac{1}{z_j-\lambda}\right)^{(r)}\right| \leqslant$$
$$\leqslant 2r!\,\frac{1}{|z_j-\lambda|^{r+1}} \leqslant \frac{2r!}{(2\delta)^p}\,\frac{(2\beta_j)^p}{|z_j-\lambda|^{r+1}}, \quad p \geqslant 0.$$

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Consequently, for s - k $\geqslant$ 1

$$\left|\left(\sum_j \frac{2|\beta_j|}{(\alpha_j-\lambda)^2+\beta_j^2}\right)^{(s-k)}\right| \leqslant \frac{2\cdot(s-k)!}{(2\delta)^{\frac{s-k+1}{2}}} \sum_j \frac{(2|\beta_j|)^{\frac{s-k+1}{2}}}{|z_j-\lambda|^{s-k+1}} \leqslant$$
$$\leqslant \frac{2\cdot(s-k)!}{2\delta^{\frac{s-k+1}{2}}}\left(\sum_j \frac{2|\beta_j|}{|z_j-\lambda|^2}\right)^{\frac{s-k+1}{2}} =$$
$$= 2\,\frac{(s-k)!}{(2\delta)^{\frac{s-k+1}{2}}}\,|\chi'(\lambda)|^{\frac{s-k+1}{2}}. \quad (6.14)$$

Let at point λ be fulfilled the inequality $|\chi'(\lambda)| \geqslant 1$. Let us assume besides the fact that at this same point for all k = 2, 3, ..., s

$$|\chi^{(k)}(\lambda)| \leqslant L^k \cdot k!\,|\chi'(\lambda)|^k, \quad (6.15)$$

where L - certain constant, and let us demonstrate that then (6.15) remains in force also for k = s + 1. For this purpose, we substitute inequalities (6.15)

in (6.13); using inequality (6.14) and by agreement $|\chi'(\lambda)| \geqslant 1$, let us find

$$|\chi^{(s+1)}(\lambda)| \leqslant L^s s! |\chi'(\lambda)|^{s+1} + \sum_{k=0}^{s-1} s! \frac{2L^k}{(2\delta)^{\frac{s-k+1}{2}}} |\chi'(\lambda)|^s \leqslant$$
$$\leqslant s! |\chi'(\lambda)|^{s+1} \left(L^s + \sqrt{\frac{2}{\delta}} \left(L^s + (2\delta)^{-s/2}\right)\right) \leqslant$$
$$\leqslant L^{s+1} (s+1)! |\chi'(\lambda)|^{s+1},$$

if constant L is selected by sufficiently large (for example, if $L > (1 + 2\sqrt{2/\delta})$.

Further, from (6.13) and (6.14) follows that at those points, where $|\chi'(\lambda)| \geqslant 1$, knowingly

$$|\chi''(\lambda)| \leqslant \left(1 + \frac{1}{\delta}\right) |\chi'(\lambda)|^2 < L^2 \cdot 2! |\chi'(\lambda)|^2,$$

if constant L is great. Therefore in all points λ, where $|\chi'(\lambda)| > 1$, and for all k $\geqslant$ 2 are fulfilled inequalities (6.15), constant L in them on λ not depending.

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Let us show now that the assumption $\sup |\chi'(\lambda)| = \infty$ leads to contradiction with lemma 17. Let us select the sequence of points λ_k

so that $M_k = |\chi'(\lambda_k)| \to \infty$, and let us assume $t_k = \frac{1}{4M_kL}$.

In accordance with (6.8) with k → ∞

$$\int_{\lambda_k - t_k}^{\lambda_k} \chi(\lambda)\, d\lambda - \int_{\lambda_k}^{\lambda_k + t_k} \chi(\lambda)\, d\lambda = o(t_k). \qquad (6.16)$$

With large k $|t_k| < \delta/2$, and on lemma 18 analytic in range $|\lambda - \lambda_k| \leqslant t_k$ function $\chi(\lambda)$ is decompose/expanded there in Taylor series:

$$\chi(\lambda) = \chi(\lambda_k) + \chi'(\lambda_k)(\lambda - \lambda_k) + \sum_{s=2}^{\infty} \frac{(\lambda - \lambda_k)^s}{s!} \chi^{(s)}(\lambda_k).$$

By substituting this expansion into right part (6.16), we will obtain with the help of estimation (6.15) the following inequality:

$$\begin{aligned} &\left| \int_{\lambda_k - t_k}^{\lambda_k} \chi(\lambda)\, d\lambda - \int_{\lambda_k}^{\lambda_k + t_k} \chi(\lambda)\, d\lambda \right| = \\ &= \left| \chi'(\lambda_k)\, t_k^2 + 2 \sum_{s=2}^{\infty} \frac{t_k^{2s}}{2s!} \chi^{(2s-1)}(\lambda_k) \right| \geqslant \\ &\geqslant |\chi'(\lambda_k)\, t_k^2| \left| 1 - 2 \sum_{s=2}^{\infty} L^s t_k^s |\chi'(\lambda_k)|^s \right| \geqslant \\ &\geqslant \frac{t_k^2}{3} |\chi'(\lambda_k)| = \frac{1}{12L}\, t_k \neq o(t_k). \end{aligned}$$

The obtained contradiction proves the need for condition $\Gamma \in \Lambda$. Theorem 6 is completely demonstrated.

Observation. The incidentally proved following equality, which follows from (6.2) and (6.12):

$$\lim_{\tau \to 0} (\rho(\tau))^{1/\tau} = e^{-\delta}, \quad \delta = \inf_{\beta_j \neq 0} |\beta_j|.$$

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Chapter VII

FILTRATION AND THE EVALUATION OF THE AVERAGE VALUE.

§1. Best unbiased estimates.

1. Formulation of the problem. Let us examine the random process of the form

$$\xi(t)=\theta(t)+\Delta(t), \quad t\in T, \qquad (1.1)$$

where $\theta(t),\ t\in T$ - the unknown determined function from a defined class Θ, a Δ (t), $t\in T$ are a Gaussian stationary process with the zero average and certain correlation function B (t).

For a clarity it is possible to interpret $\theta = \theta$ (t) as useful signal, a $\Delta = \Delta$ (t) - as the appearing in corresponding communication channel random noise. Us will interest the problem of the optimum filtration of random process $\xi = \xi$ (t) - isolation/liberation from

(1.1) unknown signal $\theta \in \Theta$. This problem includes in that in order to convert the "entering signal" ξ for the purpose as it is possible more precisely to reproduce θ. Specifically, the obtained after transform random process $\hat{\theta} = \hat{\theta}$ (t) must satisfy the following requirements: first, the average value of difference $\hat{\theta}$ (t) - θ (t) must be equal to zero:

$$M[\hat{\theta}(t) - \theta(t)] = 0, \quad t \in T, \qquad (1.2)$$

a in the second place, the RMS value of this difference must be minimum:

$$M[\hat{\theta}(t) - \theta(t)]^2 = \min. \qquad (1.3)$$

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The probability distribution of the random process of form (1.1) depends on the functional parameter $\theta \in \Theta$, so that there is a family of distributions P_θ, where the parameter $\theta = \theta$ (t), $t \in T$, is the average value of the corresponding distribution P_θ:

$$\theta(t) = M_\theta \xi(t), \quad t \in T. \qquad (1.4)$$

In this case correlation function is one and the same with all θ, namely:

$$M_\theta[\xi(s) - \theta(s)][\xi(t) - \theta(t)] = B(t - s) \qquad (1.5)$$

(here and subsequantly $M_\theta \eta$ will indicate the mathematical expectation of random variable η, calculated according to corresponding probability distribution P_θ).

The problem of filtration, actually, coincides with the problem of the determination of best estimator $\hat{\theta} = \hat{\theta}$ (t) unknown average $\theta = \theta$ (t). Special interest this problem is of in the case, when $\theta = \theta$ (t) takes the form

$$\theta(t) = \sum_{k=1}^{N} \alpha_k \theta_k(t), \qquad (1.6)$$

where $\theta_1(t), \ldots, \theta_N(t)$ - some assigned functions, and $\alpha_1, \ldots, \alpha_N$ - unknown (real) coefficients ("regression coefficients").

Each of these coefficients is linear functional of $\theta \in \Theta$. Let us say, if in the linear space of all functions $\theta = \theta$ (t) form (1.6) is determined certain scalar product (θ', θ''), and to function $\theta_1, \ldots, \theta_N$ is formed base in this space, then

$$\alpha_k = (\theta_k^*, \theta), \quad k = 1, \ldots, n, \qquad (1.7)$$

where $\theta_1^*, \ldots, \theta_N^*$ - the conjugated/combined set of functions, determined by the conditions

$$(\theta_k^*, \theta_j) = \begin{cases} 1 \ (1)\text{при} & j = k, \\ 0 \ (1)\text{при} & j \neq k, \end{cases} \quad j, k = 1, \ldots, N.$$

Key: (1). with.

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As it will be shown subsequently, the best unbiased estimates $\hat{\theta}$ (t), $t \in T$, representable in the form

$$\hat{\theta}(t) = \sum_{k=1}^{N} \hat{\alpha}_k \theta_k(t), \quad t \in T, \qquad (1.8)$$

where $\hat{\alpha}_1, \ldots, \hat{\alpha}_N$ - the best unbiased estimates for unknown coefficients $\alpha_1, \ldots, \alpha_N$.

Linear functionals of $\theta \in \Theta$ are the separately undertaken values θ (t). Subsequently will be examined the question concerning the best *unbiased estimates for arbitrary linear functionals* $\alpha = \alpha$ (θ) of unknown function $\theta = \theta$ (t), $t \in T$. In the study of this problem it is logical to suppose that all the probability distributions P_θ are equivalent to each other. Without limiting generality, it is possible to count that they are equivalent to distribution $P = P_0$, to the corresponding zero value of the parameter θ (i.e. θ (t) $\equiv$ 0). This will be assumed subsequently.

2. Necessary and sufficient statistics. Completeness of the family of distributions. Let us refine, that statistics we call any real random variable $\eta = \eta(\omega)$, measurable relative to σ-algebra $\mathfrak{A}(T)$.

which is generated by values ξ (t) = ξ (ω, t) the "observed" random process (here $t \in T$ and $\omega \in \Omega$, where Ω is space of simple events). It is logical, arises the question concerning that, with the examinationof which statistics it is possible to be bounded when evaluating the unknown parameter $\theta \in \Theta$ without the loss of any information.

Speaking about one family or the other statistician $\eta = \eta(\omega)$, it seems expedient to be converted directly to generated them σ-algebra $\mathfrak{B}$. In connection with this let us call σ-algebra $\mathfrak{B} \subseteq \mathfrak{A}(T)$ sufficient for the evaluated parameter $\theta \in \Theta$, if the conditional probabilities calculated for all events $P_\theta(A/\mathfrak{B})$, according to corresponding $A \in \mathfrak{A}(T)$ distributions P_θ, are such, that with each $\theta \in \Theta$ to probability 1

$$P_\theta(A/\mathfrak{B}) = P(A/\mathfrak{B}) \qquad (1.9)$$

(i.e. speaking in general terms, the conditional probabilities $P_\theta(A/\mathfrak{B})$ do not depend on the unknown parameter $\theta \in \Theta$).

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Let us call σ-algebra $\mathfrak{B}$ necessary, if it is contained in any sufficient σ-algebra (perhaps, supplemented by the sets of probability 0) [1].

FOOTNOTE [1]. Comp., for example [17], pp. 33-37. ENDFOOTNOTE.

Let us note that if $\mathfrak{B}$ is sufficient σ-algebra, then for any real value (having mathematical expectation $M_\theta\eta$ with each $\theta\in\Theta$) valid is the following formula: with probability 1:

$$M_\theta(\eta/\mathfrak{B}) = M(\eta/\mathfrak{B}), \quad \theta\in\Theta. \qquad (1.10)$$

This immediately follows from equality (1.9), if one considers that the conditional mathematical expectation $M_\theta(\eta/\mathfrak{B})$ can be defined as limit

$$M_\theta(\eta/\mathfrak{B}) = \lim_{n\to\infty}\sum \frac{k}{n} P_\theta\left(\frac{k-1}{n}\leqslant\eta<\frac{k}{n}\Big/\mathfrak{B}\right),$$

where there is in form mean convergence with P_θ; on the strength of the equivalency of distributions P_θ this limit with probability 1 one and the same during any $\theta\in\Theta$.

Let

$$p_\theta(\omega) = P_\theta(d\omega)/P(d\omega) \qquad (1.11)$$

be density of distribution P_θ. Let us designate $\mathfrak{B}$ σ-algebra of events, generated by all values $p_\theta(\omega)$ from $\omega\in\Omega$ $(\theta\in\Theta)$.

Lemma 1. σ-algebra $\mathfrak{B}$ indicated is sufficient for the parameter $\theta\in\Theta$.

Proof. In fact, with any $A \in \mathfrak{A}(T)$ and $B \in \mathfrak{B}$

$$P_\theta(AB) = \int_{AB} p_\theta(\omega)\,\mathsf{P}(d\omega) = \mathsf{M}[\mathsf{M}(\chi_A\chi_B p_\theta/\mathfrak{B})] =$$
$$= \mathsf{M}[\chi_B p_\theta \mathsf{M}(\chi_A/\mathfrak{B})] = \int_B \mathsf{P}(A/\mathfrak{B})\,p_\theta(\omega)\,\mathsf{P}(d\omega)$$

is simultaneous

$$\mathsf{P}_\theta(AB) = \int_B \mathsf{P}_\theta(A/\mathfrak{B})\,\mathsf{P}_\theta(d\omega) = \int_B \mathsf{P}_\theta(A/\mathfrak{B})\,p_\theta(\omega)\,\mathsf{P}(d\omega),$$

where $\chi_A = \chi_A(\omega)$ and $\chi_B = \chi_B(\omega)$ - the indicators of sets A and B.

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Since B is an arbitrary multitude from $\mathfrak{B}$, for measurable relative to $\mathfrak{B}$ integrands we have with almost all $\omega \in \Omega$

$$\mathsf{P}_\theta(A/\mathfrak{B})\,p_\theta(\omega) = \mathsf{P}(A/\mathfrak{B})\,p_\theta(\omega),$$

but $p_\theta(\omega)$, as density of equivalent measures P_θ and P, is almost everywhere positive, so that with probability 1 occurs equality (1.9).

Let us note that all values p_θ, $\theta \in \Theta$, determined formula (1.11),

satisfy condition $Mp_0 = 1$. Let us designate L^1 the space of all measurable relative to σ-algebra $\mathfrak{B}$ values η, for which $M|\eta| < \infty$.

The family of distributions P_0, $0 \in \Theta$, is called limited full/total/complete in L^1, if for any limited value η (measurable relative to $\mathfrak{B}$)) the relationship/ratio

$$M_0\eta = M\eta p_0 = 0, \quad \theta \in \Theta,$$

is equivalent to the fact that $\eta = 0$ with probability 1.

The property of the limited completeness indicates [1] that any linear continuous functional on banach space L^1 (with norm $\|\eta\| = M|\eta|$), that turns in 0 on cell/elements $p_0 \in L^1$, is equal to 0 identically.

FOOTNOTE 1. See, for example, [26], page 35. ENDFOOTNOTE.

Thus, the property of the limited completeness of the family of distributions P_0, $0 \in \Theta$, is equivalent to the fact that the linear closure of values p_0, $0 \in \Theta$, is everywhere dense in L^1.

Lemma 2. For limited full/total/complete family of distributions P_0, $0 \in \Theta$, above σ-algebra $\mathfrak{B}$ (generated by all values of form (1.11)

indicated) is not only sufficient, but also necessary.

Proof. In fact, if $\mathfrak{B}$ is not necessary, then it contains certain sufficient σ-algebra $\mathfrak{G}$ such, that certain multitude $B \in \mathfrak{B}$ does not enter in addition/completion $\mathfrak{G}^*$, in other words, value $\eta = \eta(\omega)$ form

$$\eta(\omega) = \begin{cases} 1 \text{ при } \omega \in B, \\ 0 \text{ при } \omega \notin B \end{cases}$$

Key: (1). with.

Is not measurable relatively $\mathfrak{G}^*$.

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Since $\mathfrak{G}$ is sufficient σ-algebra, value $\hat{\eta} = M_0(\eta/\mathfrak{G})$ does not depend on $\theta \in \Theta$ and $M_0(\eta - \hat{\eta}) = 0$ with all $\theta \in \Theta$. Further, once $0 \leqslant \eta \leqslant 1$, then also $0 \leqslant \hat{\eta} \leqslant 1$, so that value $\Delta = \eta - \hat{\eta}$ is limited. On the strength of the completeness of family P_0, $\theta \in \Theta$, limited (measurable relatively $\mathfrak{B}$) value $\Delta = \eta - \hat{\eta}$ are s with probability 1 is equal to 0 and $\eta = \hat{\eta}$. Thus, value η in reality is measurable relative to σ-algebra $\mathfrak{G}^*$. The obtained contradiction shows that σ-algebra $\mathfrak{B}$ is necessary.

For by us the Gaussian distributions P_0 in question the conditions of equivalency and the explicit form of values (1.11) were found earlier. Specifically,, according to theorem 7 chapters III for the equivalency of Gaussian distributions P_0 and P necessary is sufficient in order that the average value $\theta(t)$, $t \in T$, would belong to

space Y of all real functions y (t), $t \in T$, representable in the form

$$y(t) = \int e^{-i\lambda t} \varphi(\lambda) F(d\lambda), \quad t \in T, \qquad (1.12)$$

where $\varphi(\lambda)$ - certain function from appropriate space $L_T(F)$, and F (dλ) - the spectral measure of the random process ξ (t), stationary with respect to probability distribution P.

As already mentioned (see §6 chapter I), representation (1.12) singularly. Let us introduce on the linear space Y scalar product, after placing

$$\langle y_1, y_2 \rangle = \langle \varphi_1, \varphi_2 \rangle_F, \qquad (1.13)$$

where $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$ - cell/elements from the hilbert spaces $L_T(F)$, which correspond by formula (1.12) to functions $y_1 = y_1(t)$ and $y_2 = y_2(t)$. Thus, for equivalent distributions P_θ, $\theta \in \Theta$, a parametric multitude Θ is set in hilbert space Y:

$$\Theta \subseteq Y. \qquad (1.14)$$

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In order to clarify the naturalness of insertion Θ in Y, let us point out in the cases, when the set T is final, so that there is a finite number of Gaussian values $\xi(t_1), \ldots, \xi(t_n)$ with unknown average values $\theta(t_1), \ldots, \theta(t_n)$ and one and the same correlation matrix/die. If this matrix/die is nondegenerate, then Y is the n-dimensional vector

space, in which θ there is the determined multitude of vectors θ with coordinates $\theta(t_1), \ldots, \theta(t_n)$. If the correlation matrix/die is degenerated and has rank m < n, then Y is m-dimensional subspace in the n-dimensional vector space, on which is concentrated each of the degenerate Gaussian distributions P_θ (probability distributions of values $\xi(t_1), \ldots, \xi(t_n)$).

According to formula (3.2) chapter of III value (1.11) they take the form

$$p_\theta(\omega) = C_\theta \exp\{\eta_\theta(\omega)\}, \qquad (1.15)$$

where

$$C_\theta = \exp\left\{-\frac{1}{2} M\eta_\theta^2\right\}, \quad \eta_\theta = \int \varphi_\theta(\lambda)\,\Phi(d\lambda)$$

and $\Phi(\lambda)$ it indicates the stochastic spectral measure of stationary with respect to distribution P Gaussian process ξ (t), a $\varphi_\theta(\lambda)$ is that function from space $L_T(F)$, which figures in representation (1.12) of the corresponding function $\theta \in Y$:

$$\theta(t) = \int e^{-i\lambda t}\varphi_\theta(\lambda)\,F(d\lambda), \quad t \in T. \qquad (1.16)$$

Lemma 3. If a parametric multitude $\Theta \subseteq Y$ contains certain parallelepiped from its locked linear closure $\bar{\Theta}$, then the family of distributions P_θ is limited fully.

Proof. In Hilbert subspace $\bar{\Theta} \subseteq Y$ with certain orthonormalized base $\theta_1, \theta_2, \ldots$ by parallelepiped we understand many all points $\theta = \sum c_k \theta_k$, which have the finite number different from 0 coordinates $C_1, C_2, \ldots$, which satisfy the condition

$$|c_k - c_k^0| \leqslant \delta_k, \quad k = 1, 2, \ldots$$

where $c_1^0, c_2^0,$ - the coordinates of certain fixed/recorded cell/element, and $\delta_1, \delta_2, \ldots$ - some positive numbers. Let us assume $\eta_k = \eta_{\theta_k}$ (see formula (1.15)), k = 1, 2, ... To them (1.16) it is evident that

$$\eta_\theta = \sum c_k \eta_k \,(1)\, \text{при} \quad \theta = \sum c_k \theta_k.$$

Key: (1). with.

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It is easy to see that values $p_\theta, \theta \in \Theta$, determined by formula (1.15) (in which $\eta_\theta, \theta \in \Theta$ - Gaussian values from space H (T)), enter in hilbert space $H(t)$. It is clear that the system $p_\theta, \theta \in \Theta$, will be full/total/complete in space L^1, if it there is completely in Hilbert space L^2 (designated subsequently simply L), which consists of all values $\eta \in H(T)$, measurable relative to σ-algebra $\mathfrak{B}$ (recall that $\mathfrak{B}$ it is generated by system $p_\theta, \theta \in \Theta$, or, which is the same thing, by the system of values $\eta_\theta, \theta \in \Theta$).

Let $\varphi = \varphi(y_1, \ldots, y_n)$ be a Borel function on N-dimensional real

space. It is obvious, the set of all values $\eta \in L$ form $\eta = \varphi(\eta_1, \ldots, \eta_n)$ forms everywhere dense set in all space L. and during fixed/recorded n it forms the locked space.

let us assume temporarily that at any n the system of values p_0 form (1.15) (where $\theta = \sum_{k=1}^{n} c_k \theta_k$, $|c_k| \leqslant \delta_k$, $k = 1, \ldots, n$, and $\delta_1, \ldots, \delta_n$ - some positive numbers, a $\eta_0 = \sum_{k=1}^{n} c_k \eta_k$) is everywhere dense in the subspace, formed by all values $\eta = \varphi(\eta_1, \ldots, \eta_n)$. Then, obviously, the linear closure of all values p_0, $\theta \in \Theta$, will be everywhere dense in all space L. Therefore it suffices to demonstrate only that if

$$\mathsf{M}\eta\, p_0 = \mathsf{M}_0 \eta = C_0 \int_{R^n} \varphi(y_1, \ldots, y_n) \exp\left\{\sum_{k=1}^{n} c_k y_k\right\} \mathsf{P}(dy) = 0$$

with all $\theta = \sum_{k=1}^{n} c_k \theta_k$ (where $\mathsf{P}(dy)$ is distribution of values $\eta_1, \ldots, \eta_n$ in n-dimensional space), then also function itself $\varphi(y_1, \ldots, y_n)$ is equal to 0 almost everywhere relatively $\mathsf{P}(dy)$. But this is well known fact [1], since the densities

$$\frac{\mathsf{P}_0(dy)}{\mathsf{P}(dy)} = C_0 \exp\left\{\sum_{k=1}^{n} c_k y_k\right\},$$

where $|c_k| \leqslant \delta_k$, $k = 1, \ldots,$ n, form exponential family.

FOOTNOTE [1]. See, for example, [17], page 76. ENDFOOTNOTE.

Thus, lemma is demonstrated.

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3. Unbiased estimates. Formulas (1.12), (1.13) give the isometric conformity of hilbert spaces Y, $L_T(F)$ and H (T), with which

$$y \in Y \leftrightarrow \varphi \in L_T(F) \leftrightarrow \eta \in H(T) \qquad (1.17)$$
$$\langle y_1, y_2\rangle = \langle \varphi_1, \varphi_2\rangle_F = \langle \eta_1, \eta_2\rangle.$$

Here H (T), as before it indicates the locked linear closure of values ξ (t), $t \in T$, the stationary with respect to distribution P Gaussian process, in which the scalar product is $\langle \eta_1, \eta_2\rangle = M\eta_1\eta_2$, $\Phi(d\lambda)$ - the corresponding stochastic spectral measure, and value η in (1.17) is connected with functions φ (λ) and y (t) (figuring in (1.12)) by the equality

$$\eta = \int \varphi(\lambda)\,\Phi(d\lambda). \qquad (1.18)$$

Let us note that since

$$B(s-t) = \int e^{-i\lambda t} e^{i\lambda s} F(d\lambda),$$

with any s the function $y_s(t) = B(s-t)$ of $t \in T$ enters in space Y, whereupon the system of all functions y_s, $s \in T$, it is full/total/complete. Completeness follows, in particular, from that fact that full/total/complete is the set of functions $e^{i\lambda s} \in L_T(F)$.

which correspond to functions $y_s \in Y$:

$$y_s(t) = B(s-t) \leftrightarrow e^{i\lambda s} \leftrightarrow \xi(s).$$

According to formula (1.16) each functional of the functional parameter $\theta = \theta$ (t), $t \in T$, defined as value of function $\theta = \theta$ (t) at certain fixed/recorded point $t \in T$, let us present in the form

$$\theta(t) = \langle e^{i\lambda t}, \varphi_\theta(\lambda)\rangle_F = \langle y_t, \theta\rangle. \qquad (1.19)$$

It is evident that the functional θ (t) of $\theta \in \Theta$ is continued into linear continuous functional on hilbert space Y, whereupon unambiguously to the subspace $\bar{\Theta} \subseteq Y$, which is the locked linear closure of cell/elements $\theta \in \Theta$.

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Below we will consider the unbiased estimates for the arbitrary functionals $\alpha = \alpha$ (θ) of $\theta \in \Theta$, which are the contraction on $\Theta \subseteq Y$ of certain linear continuous functional in hilbert space Y, representable by the common/general/total formula of the form

$$\alpha(\theta) = \langle y, \theta\rangle, \quad \theta \in \Theta, \qquad (1.20)$$

where y = y (t), $t \in T$, certain function from space Y.

It is useful to note that the functional α (θ) of $\theta \in \Theta$

allow/assumes the linear unbiased estimate (i.e. this estimation $\hat{\alpha} \in H(T)$, **for which** $M_\theta\hat{\alpha} = \alpha(\theta)$, $\theta \in \Theta$) when and only when it is the contraction on $\Theta \subseteq Y$ of certain linear continuous functional in hilbert space Y. Specifically, for the functional of form (1.19) the linear unbiased estimate is value ξ (t).

Actually, taking into account that which was described in (1.17) unitary isomorphism, from formula (1.22), which is derive/concluded below, for the functional of form (1.20) we obtain

$$\alpha(\theta) = \langle y, \theta \rangle = \langle \eta, \eta_0 \rangle = M_\theta \eta \textbf{ with } \theta \in \Theta, \quad (1.21)$$

where $\eta \in H(T)$ - the value of form (1.18), that corresponds to function $y \in Y$. On the other hand, each value $\eta \in H(T)$, representable in the form $\eta = \int \varphi(\lambda) \Phi(d\lambda)$, answers functional α (θ) form (1.21), allow/assume continuation from set $\Theta \subseteq Y$ whole space Y. In this case its assigning function $y \in Y$ is determined by the formula

$$y(t) = \int e^{-i\lambda t} \varphi(\lambda) F(d\lambda), \quad t \in T.$$

Lemma 4. Any value $\eta \in H(T)$ has the final mathematical expectation $M_\theta \eta$, whatever $\theta \in \Theta$. In this case for values η from subspace H (T), generated as values ξ (t), $t \in T$, occurs the following formula:

$$M_\theta\eta = \langle \eta, \eta_\theta \rangle, \quad \theta \in \Theta. \qquad (1.22)$$

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Proof. Any value $\eta \in H(T)$, as limit of the Gaussian values of form $\sum c_k \xi(t_k)$, itself is Gaussian. Therefore each of values $\eta_0 \in H(T)$, which figure in formula (1.15), is Gaussian, so that density $p_0 = p_0(\omega)$ is integrated squared with P (in other words, value $p_0 = p_0(\omega)$ enters in hilbert space $H(T)$). For any limited value $\eta \in H(T)$ we have

$$M_0\eta = M\eta p_0 = \langle \eta, p_0 \rangle. \qquad (1.23)$$

For any value $\eta \in H(T)$ it is possible to select that which is converging to η (in hilbert space $H(T)$) the sequence of the limited values $\eta_1, \eta_2, \ldots,$ which is fundamental, also, in the sense of mean convergence with P_0:

$$M_0|\eta_m - \eta_n| = \langle |\eta_m - \eta_n|, p_0 \rangle \leq \|\eta_m - \eta_n\| \cdot \|p_0\| \to 0$$

with m, n $\to \infty$. It is obvious, sequence $\eta_1, \eta_2, \ldots$ converge in mean with P_0 precisely to value η, since P_0 is equivalent to measure P. Consequently, $M_0|\eta| < \infty$, whereupon for a mathematical expectation $M_0\eta$ is valid formula (1.23). Further, for values ξ (t), $t \in T$, simultaneously occur the following equalities (see (1.4), (1.15) and (1.16)):

$$\theta(t) = M_0\xi(t) = \int e^{-i\lambda t}\varphi_0(\lambda) F(d\lambda) \doteq \langle \xi(t),\ \eta_0\rangle.$$

From the obtained equation

$$M_0\eta = \langle \eta, \eta_0\rangle \quad \textbf{with} \quad \eta = \xi(t), \quad t \in T,$$

in an obvious manner follows that the mathematical expectation $M_0\eta$ of $\eta \in H(T)$, ~~was~~ linear continuous functional of form (1.22) on hilbert space $H(T)$, on subspace H (T) (generated by values ξ (t), $t \in T$) can be registered in the form

$$M_0\eta = \langle \eta,\ p_0\rangle = \langle \eta,\ \eta_0\rangle.$$

It is evident also that the figuring in formula (1.15) value $\eta_0 \in H(T)$ coincides with the projection of value $p_0 \in H(T)$ on subspace H (T).

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Lemma is demonstrated.

Let us designate $\mathfrak{B}$ σ-algebra, generated by all values $\eta_0 = \eta_0(\omega)$ on Ω $(\theta \in \Theta)$; designate L subspace into the Hilber space $H(T)$, formed by all values $\eta \in H(T)$, which are measurable relative to σ-algebra $\mathfrak{B}$, and L - the locked linear closure of all values η_0 $(\theta \in \Theta)$.

As has already been indicated, from formulas (1.15), (1.16) it follows that

$$\eta_0 = \sum c_k \eta_{0_k} \quad \textbf{with} \quad \theta = \sum c_k \theta_k,$$

and, consequently (see formula (1.22)), for any value $\eta \in H(T)$

$$M_0\eta = \langle \eta, \eta_0 \rangle = \sum_k c_k \langle \eta, \eta_{0_k} \rangle = \sum c_k M_{0_k}\eta.$$

In view of this it is clear that for any functional α (θ) form (1.20) (see also (1.21)), assigned on an arbitrary parametric multitude $\Theta \subseteq Y$, its linear unbiased estimate $\eta \in H(T)$ simultaneously is the linear unbiased estimate for linear continuation α (θ) to the locked linear closure $\bar{\Theta}$ an initial multitude Θ. This indicates that in the examination of the linear unbiased estimates, without limiting generality, it is possible to count

$$\Theta = \bar{\Theta}. \qquad (1.24)$$

Under this assumption we deal with full/total/complete family of distributions P_0, $0 \in \Theta$ (see lemma 3), and necessary and sufficient σ-algebra for the parameter $0 \in \Theta$ it is σ-algebra $\mathfrak{B}$, generated values $\eta_0 = \eta_0(\omega)$ from formula (1.15), i.e., the values of the form

$$\eta_0 = \int \varphi_0(\lambda)\,\Phi(d\lambda), \qquad (1.25)$$

where the function $\varphi_0(\lambda) \in L_T(F)$ they are determined from integral equations (1.16):

$$\theta(t) = \int e^{-i\lambda t}\varphi_0(\lambda)\,\Phi(d\lambda), \quad t \in T$$

(parameter θ passes here certain full/total/complete system in Θ).

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Let α (θ) be a functional of the unknown parameter $\theta \in \Theta$, described by formula (1.20), and $\eta \in H(T)$ - certain linear unbiased estimate (see (1.21)). Let us assume

$$\hat{\alpha} = \mathsf{M}(\eta/\mathfrak{B}). \qquad (1.26)$$

Since σ-algebra $\mathfrak{B}$ is sufficient for the parameter $\theta \in \Theta$, is simultaneous

$$\hat{\alpha} = \mathsf{M}_\theta(\eta/\mathfrak{B}), \quad \theta \in \Theta,$$

and

$$\mathsf{M}_\theta\hat{\alpha} = \mathsf{M}_\theta\eta = \alpha(\theta) \text{ (1) при всех } \theta \in \Theta;$$

Key: (1). with all.

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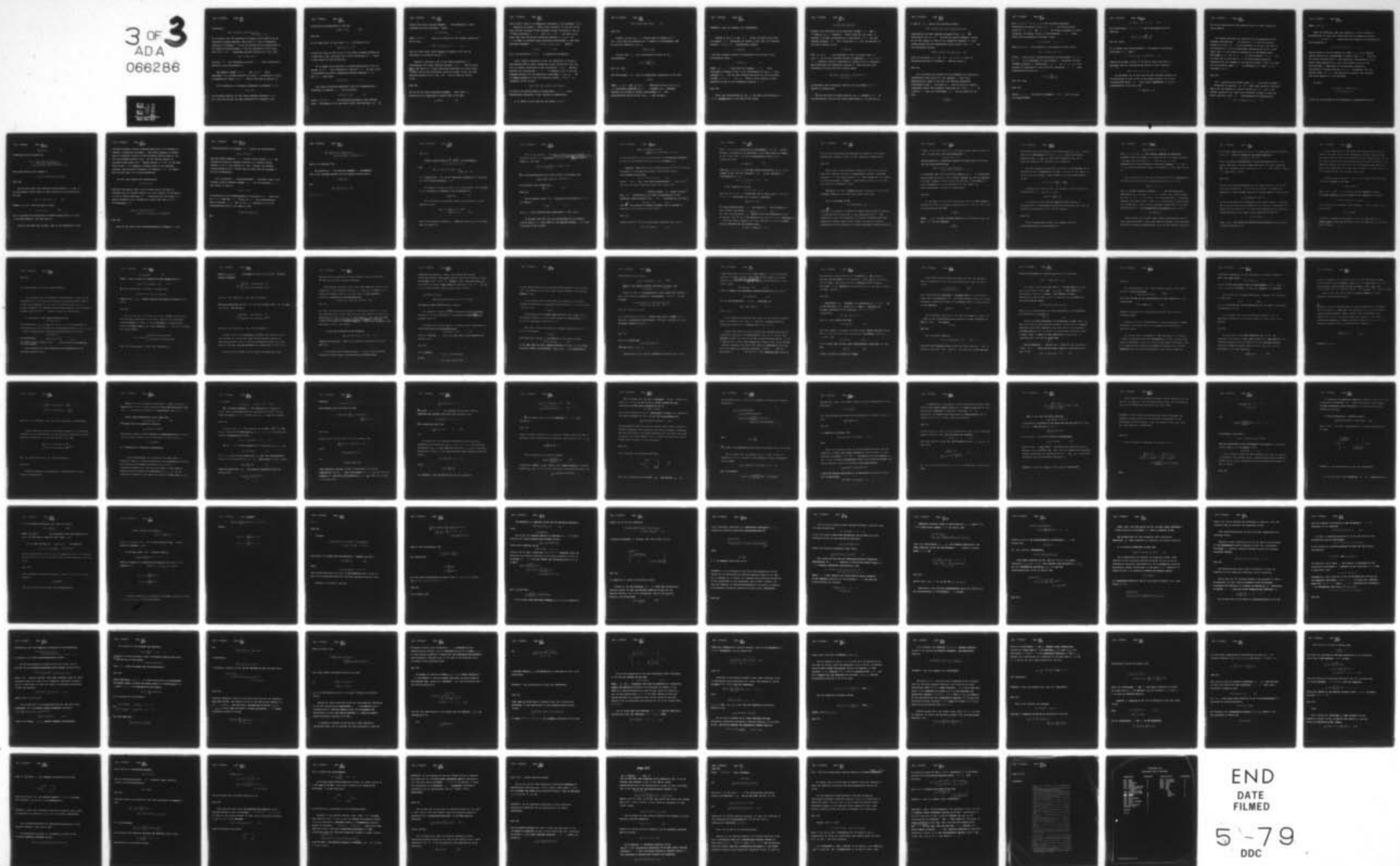

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furthermore,

$$M_\theta(\hat\alpha - \alpha(\theta))^2 = M_\theta(\eta - \alpha(\theta))^2 - M_\theta(\eta - \hat\alpha)^2 \leqslant \leqslant M(\eta - \alpha(\theta))^2. \quad (1.27)$$

It is evident that the determined by formula (1.26) value $\hat\alpha$ is the "improved" unbiased estimate. This value $\hat\alpha = \hat\alpha(\omega)$ is measurable relative to σ-algebra $\mathfrak{B}$ is (on the strength of the completeness of the family of distributions P_θ) the only estimation of this type, since if $\eta = \eta(\omega) - \mathfrak{B}$- measurable unbiased estimate for $\alpha(\theta)$, then

$$M_\theta(\hat\alpha - \eta) = M_\theta\hat\alpha - M_\theta\eta = 0$$

with all $\theta \in \Theta$, and therefore in reality $\hat\alpha = \eta$ (with probability 1 relative to any distribution P_θ).

For Gaussian values $\eta \in H(T)$ and $\eta_\theta (\theta \in \Theta)$ value indicated $\hat\alpha = M(\eta/\mathfrak{B})$ is the projection of value η on subspace L, which is generated by values $\eta_\theta \in H(T)$ form (1.25) (see §5 chapter I).

Let us designate L_0 orthogonal complement to subspace $L \subseteq H(T)$:

$$H(T) = L \oplus L_0.$$

As we recently showed, all the linear unbiased estimates $\eta \in H(T)$ for $\alpha(\theta)$ have one and the same projection $\hat\alpha$ on subspace L and

therefore are representable in the form

$$\eta = \hat{\alpha} \oplus \Delta, \quad \Delta \in L_0. \qquad (1.28)$$

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On the other hand, for any value $\Delta \in L_0$ by formula (1.22)

$$M_\theta \Delta = \langle \Delta, \eta_\theta \rangle = 0, \quad \theta \in \Theta,$$

so that any value of form (1.28) is the linear unbiased estimate for a functional α (θ), whereupon (see (1.27)) estimation $\hat{\alpha} \in L$ there is best among all such estimations.

It is useful to be converted to relationship/ratios (1.17). To values $\eta_\theta \in H(T)$ they correspond to function $\theta \in Y$, so that (1.17) it determines the unitary isomorphism between subspaces $L \subseteq H(T)$ and $\Theta \subseteq Y$, with which

$$\eta_\theta \in L \leftrightarrow \theta \in \Theta. \qquad (1.29)$$

The linear continuous functional α (θ) of θ unambiguously is determined on subspace $\Theta \subseteq Y$ by the formula

$$\alpha(\theta) = \langle \theta_\alpha, \theta \rangle, \quad \theta \in \Theta, \qquad (1.30)$$

where $\theta_\alpha = \theta_\alpha(t)$, $t \in T$, the assigning functional α (θ) function from Θ. By formula (1.17) (see also (1.21)) this function $\theta_\alpha \in \Theta$ it

answers the linear unbiased estimate η from subspaces L, which coincides with best estimator $\hat{\alpha}$. Thus,

$$\hat{\alpha}=\int \varphi_\alpha(\lambda)\,\Phi(d\lambda), \tag{1.31}$$

where $\varphi_\alpha(\lambda)\in L_T(F)$ there is a solution to the integral equation of form (1.16):

$$\theta_\alpha(t)=\int e^{-i\lambda t}\varphi_\alpha(\lambda)\,F(d\lambda),\quad t\in T, \tag{1.32}$$

with the "left side", which figures in formula (1.30) for the functional α in question (θ) of $\theta\in\Theta$.

Theorem 1. Function α (θ) of the unknown parameter $\theta\in\Theta$ allow/assumes the linear unbiased estimate $\eta\in H(T)$ when and only when α (θ) there is a linear continuous functional on hilbert space $\Theta\subseteq Y$. Let α (θ) be this functional, given by formula (1.30). The best unbiased estimate $\hat{\alpha}$ for α (θ), $\theta\in\Theta$, can be found by formula (1.31).

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The set of all linear unbiased estimates η for α (θ), $\theta\in\Theta$, geometrically is hyperplane in space H (T) of the form

$$L_\alpha=\hat{\alpha}\oplus L_0 \tag{1.33}$$

(see (1.28)), where L_0 is orthogonal complement to the subspace $L \subseteq H(T)$, generated by values η_0 form (1.25). Subspace L is the set of all best unbiased estimates $\hat{\alpha}$ (for different linear functionals α (θ) of $\theta \in \Theta$). The estimations $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ for $\alpha_1(\theta), \ldots, \alpha_n(\theta)$ are best in the sense that them correlation insulation blanket $\{s_{ij}\}$, $s_{ij} = M_0(\hat{\alpha}_i - \alpha_i(\theta))(\hat{\alpha}_j - \alpha_j(\theta))$, there is smallest among correlation matrix/dies $\{\sigma_{ij}\}$ all other unbiased estimates $\eta_1, \ldots, \eta_n$ $(\sigma_{ij} = M_0(\eta_i - \alpha_i(\theta))(\eta_j - \alpha_j(\theta))$, namely:

$$\{s_{ij}\} \leqslant \{\sigma_{ij}\} \qquad (1.34)$$

(i.e. the matrix/die $\{s_{ij}\} - \{\sigma_{ij}\}$ is negatively determined).

Proof. Besides inequality (1.34), all assertions of theorem 1 have demonstrated we above. Inequality (1.34) follows from that fact that the linear combination of best estimators $\hat{\alpha} = \sum c_k \hat{\alpha}_k$ belongs together with estimations $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ to subspace L and is the best unbiased estimate for the appropriate functional $\alpha(\theta) = \sum c_k \alpha_k(\theta)$ of $\theta \in \Theta$. Equate/comparing $\hat{\alpha}$ with the unbiased estimate $\eta = \sum c_k \eta_k$, on inequality (1.27) we have

$$M_\theta(\hat{\alpha} - \hat{\alpha}(\theta))^2 = \sum_{i,j} c_i c_j s_{ij} \leqslant M_0(\eta - \alpha(\theta))^2 = \sum_{ij} c_i c_j \sigma_{ij}.$$

In view of the arbitrariness of coefficients $c_1, \ldots, c_n$ hence escape/ensues inequality (1.34). Theorem is demonstrated.

It is useful to note that for any values $\eta_1, \eta_2 \in H(T)$

$$M_\theta(\eta_1 - M_\theta\eta_1)(\eta_2 - M_\theta\eta_2) = M\,\eta_1\eta_2. \qquad (1.35)$$

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Further, let θ_1, θ_2, ... - certain base in subspace $\Theta \subseteq Y$, i.e., this full/total/complete in Θ system of cell/elements, that for any real numbers α_1, α_2, ...

$$\left\| \sum_k \alpha_k \theta_k \right\|^2 \asymp \sum_k \alpha_k^2.$$

Let θ^*_1, θ^*_2, ... - the interconnected circuit of the cell/elements:

$$\langle \theta_k^*, \theta_j \rangle = \begin{cases} 1 \text{ при}^{(1)} & k = j, \\ 0 \text{ при} & k \neq j. \end{cases}$$

Key: (1). with.

Any cell/element $\theta \in \Theta$ can be unambiguously represented in the form

$$\theta = \sum_k \alpha_k \theta_k,$$

where $\alpha_k = \langle \theta_k^*, \theta \rangle$ and $\sum_k \alpha_k^2 \asymp \|\theta\|^2$, whereupon for any real α_1, α_2, ..., satisfying condition $\sum_k \alpha_k^2 < \infty$, a series $\sum_k \alpha_k \theta_k$ strongly descends and its sum is certain cell/element $\theta \in \Theta$. The interconnected circuit of θ^*_1, θ^*_2, ... also is base [1].

FOOTNOTE [1]. See, for example, [4]. ENDFOOTNOTE.

Theorem 2. Let $\alpha = \alpha(\theta)$, $\theta \in \Theta$, linear continuous functional on subspace $\Theta \subseteq Y$, determined by formula (1.30). Then its assigning function $\theta_\alpha = \theta_\alpha(t)$, $t \in T$, representable together

$$\theta_\alpha = \sum_k \alpha(\theta_k^*)\,\theta_k, \qquad (1.36)$$

a the best unbiased estimate $\hat{\alpha}$ representable in the form of the corresponding series

$$\hat{\alpha} = \sum_k \alpha(\theta_k^*)\,\eta_k, \qquad (1.37)$$

where $\eta_1, \eta_2, \ldots$ - value from the subspace $L \subseteq H(T)$, which correspond in formula (1.29) to the cell/elements of $\theta_1, \theta_2 \ldots$ from subspace $\Theta \subseteq Y$, are the best unbiased estimates for the functionals of form $\alpha_k^*(\theta) = \langle\theta_k, \theta\rangle$ of $\theta \in \Theta$. Series (1.37) converge in mean quadratic on each of the probability measures P_θ, $\theta \in \Theta$.

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Proof. The cell/elements θ_1, θ_2, ... are base, and function $\theta_\alpha = \theta_\alpha(t)$, $t \in T$, representable in the form of the series

$$\theta_a = \sum_k \langle \theta_a, \theta_k^* \rangle \theta_k = \sum_k \alpha(\theta_k^*) \theta_k.$$

Further, with conformity (1.29 indicated) between $\Theta \subseteq Y$ and $L \subseteq H(T)$ base $\theta_1, \theta_2, \ldots$ subspace Θ makes transition into base $\eta_1, \eta_2, \ldots$ subspace L, so that corresponding to cell/element $\theta_a \in \Theta$ the best unbiased estimate $\alpha \in L$ for a functional $\alpha = \alpha(\theta)$ representable in the form of series (1.37):

$$\alpha = \sum_k \alpha(\theta_k^*) \eta_k.$$

With $\alpha(\theta) = \langle \theta_k, \theta \rangle$ from expression (1.37) we obtain, what $\alpha = \eta_k$, i.e., η_k is the best unbiased estimate of function $\alpha_k^*(\theta) = \langle \theta, \theta_k \rangle$, $\theta \in \Theta$ $(k = 1, 2, \ldots)$. Finally, powerful convergence of series (1.37) is equivalent mean convergence (on probability measure P), which together with convergence of series from the average values

$$\sum_k \alpha(\theta_k^*) \mathsf{M}_\theta \eta_k = \sum_k \alpha(\theta_k^*) \langle \eta_k, \eta_\theta \rangle = \sum_k \alpha(\theta_k^*) \langle \theta_k, \theta \rangle = \\ = \langle \theta_a, \theta \rangle = \alpha(\theta)$$

is equivalent mean convergence quadratic on any measure P_θ, $\theta \in \Theta$. Theorem is demonstrated.

Let us note that if initial base $\theta_1, \theta_2, \ldots$ subspace $\Theta \subseteq Y$ is orthonormalized, then for any linear functionals $\alpha_1 = \alpha_1(\theta)$ and $\alpha_2 =$

α_2 (θ) of $\theta \in \Theta$ occurs the following formula:

$$M_\theta[\alpha_1 - \alpha_1(\theta)][\alpha_2 - \alpha_2(\theta)] = \langle \alpha_1, \alpha_2 \rangle = \sum_k \alpha_1(\theta_k)\alpha_2(\theta_k). \quad (1.38)$$

Specifically, the best unbiased estimates $\hat{\theta}$ (t), $t \in T$, for functionals θ (t) of $\theta \in \Theta$ form (1.19) (where parameter t passes set T) with respect to each of the probability measures P_θ form random process with the appropriate average value θ (t), $t \in T$, and the correlation function

$$M_\theta[\hat{\theta}(s) - \theta(s)][\hat{\theta}(t) - \theta(t)] = \sum_k \theta_k(s)\theta_k(t), \quad (1.39)$$

where the function $\theta_k = \theta_k(t)$ of $t \in T$, k = 1, 2, ..., form the orthonormalized base in subspace Θ function space Y.

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Let us consider the problem of the estimations of "regression coefficients" (see Section 1). Let subspace Θ have final dimensionality, equal N. Then any N of linearly independent cell/elements from Θ form base in Θ, just as any N of linearly independent values from subspace L form base in L. If $\theta_1, \ldots, \theta_N$ is a base in Θ, then any cell/element $\theta \in \Theta$ let us present in the form

$$\theta = \sum_{k=1}^{N} \alpha_k \theta_k,$$

where $\alpha_k = \langle \theta_k^*, \theta \rangle$, k = 1, ..., N, the so-called regression coefficients in diagram (1.1), a $\theta_1^*, \ldots, \theta_N^*$ are interconnected circuit to $\theta_1, \ldots, \theta_N$. Let $\eta_1, \ldots, \eta_N$ be values in subspace L, which correspond, by formula (1.29), to cell/elements $\theta_1, \ldots, \theta_N$. These values can be determined by formula (1.25):

$$\eta_k = \int \varphi_k(\lambda) \Phi(d\lambda), \quad k = 1, \ldots, n,$$

where $\varphi_k(\lambda) \in L_T(F)$ - the solution to the equation of form (1.16):

$$\theta_k(t) = \int e^{-i\lambda t} \varphi_k(\lambda) F(d\lambda), \quad t \in T.$$

Values η_k are the best unbiased estimates for functionals $\alpha_k^*(\theta) = \langle \theta_k, \theta \rangle$ of $\theta \in \Theta$. It is analogous, to cell/elements θ_k^* correspond the best unbiased estimates $\hat{\alpha}_k$ coefficients $\alpha_k = \langle \theta_k^*, \theta \rangle$, k = 1, ..., N. On the strength of isomorphism (1.29) these best estimators $\hat{\alpha}_1, \ldots, \hat{\alpha}_N$ form interconnected circuit to $\eta_1, \ldots, \eta_N$:

$$\langle \hat{\alpha}_k, \eta_j \rangle = \begin{cases} 1 & \overset{(1)}{\text{при}}\ k = j, \\ 0 & \overset{(1)}{\text{при}}\ k \neq j. \end{cases}$$

Key: (1). with.

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Values $\eta_1, \ldots, \eta_N$ form base in subspace $L \subseteq H(T)$, and if we use the representation

$$\hat{\alpha}_i = \sum_l s_{il}\eta_l \qquad (k = 1, \ldots, N), \qquad (1.40)$$

that coefficients $s_{ij}(i, j = 1, \ldots, N)$ can be determined from the equations

$$\sum_l s_{il}\langle\eta_l, \eta_k\rangle = \begin{cases} 1 & \overset{(1)}{\text{при}} \ i = k, \\ 0 & \text{при} \ i \neq k. \end{cases}$$

Key: (1). when.

It is evident that the matrix/die $\{s_{ij}\}$ is reverse to correlation matrix/die $\{\langle\eta_i, \eta_k\rangle\}$, where

$$\langle\eta_i, \eta_j\rangle = \langle\varphi_i, \varphi_j\rangle_F = \int \varphi_i(\lambda)\overline{\varphi_j(\lambda)}\, F(d\lambda), \qquad (1.41)$$
$$i, j = 1, \ldots, N.$$

Taking into account (1.40), it is easy to count also which $\{s_{ij}\}$ coincides with the correlation matrix/die of best estimators $\hat{\alpha}_1, \ldots, \hat{\alpha}_N$:

$$\mathsf{M}_\theta(\hat{\alpha}_i - \alpha_i)(\hat{\alpha}_j - \alpha_j) = \langle\hat{\alpha}_i, \hat{\alpha}_j\rangle = s_{ij}, \quad i, j = 1, \ldots, N. \ (1.42)$$

In conclusion let us note that the best unbiased estimate for the functional of form (1.19) is the average values θ (t) at the fixed/recorded point $t \in T$, being linear combination $\theta(t) = \sum_{k=1}^{N} \alpha_k\theta_k(t)$, representable in the form (1.8):

$$\theta(t) = \sum_{k=1}^{N} \hat{\alpha}_k\theta_k(t), \quad t \in T.$$

§2. On the evaluations of the average value as a whole. Method of least squares.

As already mentioned, the condition of the equivalency of all Gaussian distributions in question $P_\theta, \theta \in \Theta$, with the average values $\theta = \theta$ (t), $t \in T$, it is equivalent to the fact that each of the functions $\theta = \theta$ (t) allow/assumes representation (1.16), which means that $\theta = \theta$ (t) there is contraction of continuous linear functional $\langle \varphi, \varphi_0 \rangle_F$ of $\varphi \in L_T(F)$ on cell/elements $e^{i\lambda t} \in L_T(F)$. It is clear that in this case $\theta = \theta$ (t) allow/assumes analogous representation with respect to any spectral measure G (dλ), for which the corresponding space $L_T(G)$ is contained in $L_T(F)$ and $\|\varphi\|_F \leqslant \|\varphi\|_G$ with all $\varphi(\lambda) \in L_T(G)$.

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If A - operator from hilbert space $L_T(G)$ into hilbert space $L_T(F)$, determined by equality $A\varphi(\lambda) = \varphi(\lambda)$, then above condition indicated means that the operator B = A*A is limited in $L_T(G)$ (A * - the adjoint operator to A). Under this condition, as soon as which was noted, function θ (t), $t \in T$, allow/assumes the representation

$$\theta(t) = \int e^{-i\lambda t} \psi_0(\lambda)\, G(d\lambda), \quad t \in T, \qquad (2.1)$$

where $\psi_0(\lambda) \in L_T(G)$.

Under the condition, when this operator B = A*A is nuclear, a representation of type (2.1) occurs also for the Gaussian process ξ = ξ (t), $t \in T$ (stationary with respect to distribution P); with probability 1

$$\xi(t) = \int e^{-i\lambda t}\eta(\lambda)\, G(d\lambda), \quad t \in T \tag{2.2}$$

(see in regard to this §6 chapter I), where $\eta = \eta(\lambda)$ is a Gaussian random function with trajectories in Hilbert space $L_T(G)$. Since all the initial distributions P_0 are equivalent, representation (2.2) occurs (with probability 1), also, with respect to any of the distributions P_0, whereupon the corresponding probability distribution of random function $\eta(\lambda) \in L_T(G)$ is Gaussian with the average value $\psi_0(\lambda) \in L_T(G)$ and correlation operator, who coincides with above operator B = A*A INDICATED.

In fact,

$$\theta(t) = \mathsf{M}_0\xi(t) = \int e^{-i\lambda t}\mathsf{M}_0\eta(\lambda)\, G(d\lambda) = \\ = \int e^{-i\lambda t}\psi_0(\lambda)\, G(d\lambda), \quad t \in T,$$

so that on the strength of the uniqueness of representation (2.1)

$$M_0\eta(\lambda) = \psi_0(\lambda); \qquad (2.3)$$

furthermore (see §6 chapter I),

$$\begin{aligned} M_0[\langle\varphi, \eta\rangle_G - \langle\varphi, \psi_0\rangle_G][\langle\psi, \eta\rangle_G - \langle\psi, \psi_0\rangle_G] = \\ = M[\langle\varphi, \eta\rangle_G \langle\psi, \eta\rangle_G] = M[\eta(\varphi)\eta(\psi)] = \\ = \langle\varphi, \psi\rangle_F = \langle A\varphi, A\psi\rangle_F = \langle A^* A\varphi, \psi\rangle_G = \langle B\varphi, \psi\rangle_G, \end{aligned} \qquad (2.4)$$

where (see formula (6.9) chapter I)

$$\eta(\varphi) = \int \varphi(\lambda)\Phi(d\lambda) = \langle\varphi, \eta\rangle_G, \ \varphi \in L_T(G).$$

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Let us count below the "observed" random process $\xi = \xi(t)$, $t \in T$, by cell/element hilbert space X, which consists of the functions of the form

$$x(t) = \int e^{-i\lambda t}\varphi(\lambda)\, G(d\lambda), \ t \in T \qquad (2.5)$$

(where $\varphi(\lambda) \in L_T(G)$), with the scalar product

$$\langle x_1, x_2\rangle = \langle\varphi_1, \varphi_2\rangle_G, \qquad (2.6)$$

let us consider some evaluations of unknown average value $\theta = \theta(t)$, $t \in T$, as cell/element of the same space X.

Above it was shown (see (1.24)), that in the examination of the

unbiased estimates, without limiting generality, it is possible to consider a parametric multitude Θ the locked subspace in hilbert space Y, analogous recently to the introduced hilbert space X, but with the spectral measure F (dλ) - by the spectral measure of stationary with respect to P random process ξ = ξ (t). At the same time the set Θ, as subspace in hilbert space X, is, generally speaking, open-circuited. Moreover, the subspace $\Theta \subseteq X$ is locked when and only when it is finite-dimensional.

In fact, that assigns the representation

$$y(t) \in Y \rightarrow y(t) \in X$$

operator from hilbert space Y into hilbert space X actually it coincides with the adjoint operator A *, more precise, with operator A * and Y, during functions $y \in Y$ coinciding with the values $A^*y \in Y$, which by formula (1.17) correspond to values A*φ, where $\varphi \in L_T(F)$ it corresponds $y \in Y$:

$$y(t) = \int e^{-i\lambda t}\varphi(\lambda) F(d\lambda) = \langle e^{i\lambda t}, \varphi\rangle_F =$$
$$= \langle Ae^{i\lambda t}, \varphi\rangle_F = \langle e^{i\lambda t}, A^*\varphi\rangle_G = \int e^{-i\lambda t}[A^*\varphi(\lambda)] G(d\lambda).$$

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Thus, in the case of the closure/isolation of subspace Θ in X

(closure/isolation of subspace $A^*\Theta \subseteq X$) one-to-one representation

$$\theta(t) \in Y \xleftrightarrow{A^*} \theta(t) \in X$$

maps the locked subspace $\Theta \subseteq Y$ to the locked subspace $A^*\Theta \subseteq X$, and therefore according to Banach's theorem, is a bounded inverse operator (A *) - 1. But operator B = A*A - nuclear, and bounded inverse operator for A * exists when and only when the subspace Θ is finite-dimensional.

Let us designate $\tilde{\Theta}$ closing/shorting Θ in Hilbert space X and consider perform unbiased estimate $\eta \in X$ for the parameter $\theta \in \Theta$ with values in space X:

$$M_0\eta = \theta, \quad \theta \in \Theta.$$

the condition of the nondisplacement of estimation $\eta \in X$ means that $M_0\langle\eta, x\rangle = \langle\theta, x\rangle$ with any $x \in X$. If θ_1, θ_2, ... - the orthonormalized base in subspace $\Theta \subseteq X$, and x_1, x_2, ... - addition to it to the orthonormalized base in all space X, then

$$\eta = \sum_k \langle\eta, \theta_k\rangle \theta_k + \sum_k \langle\eta, x_k\rangle x_k$$

and

$$M_0\eta = \sum_k [M_0\langle\eta, \theta_k\rangle]\,\theta_k + \sum_k [M_0\langle\eta, x_k\rangle]\,x_k =$$
$$= \sum_k \langle\theta, \theta_k\rangle\,\theta_k + \sum_k \langle\theta, x_k\rangle\,x_k = \sum_k \langle\theta, \theta_k\rangle\,\theta_k = \theta.$$

Hence it is apparent that

the projection $\tilde{\eta}$ the unbiased estimate η on subspace $\tilde{\Theta}$ also is the unbiased estimate for the unknown average value:

$$\tilde{\eta} = \sum_k \langle\eta, \theta_k\rangle\,\theta_k$$

and

$$M_0\tilde{\eta} = \sum_k \langle\theta, \theta_k\rangle\,\theta_k = \theta, \quad \theta \in \Theta.$$

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Further, since values $\eta - \tilde{\eta}$ and $\tilde{\eta} - \theta$ are orthogonal,

$$\|\eta-\theta\|^2=\|\tilde{\eta}-\theta\|^2+\|\eta-\tilde{\eta}\|^2$$

and

$$M_\theta\|\tilde{\eta}-\theta\|^2=M_\theta\|\eta-\theta\|^2-M_\theta\|\eta-\tilde{\eta}\|^2\leqslant \leqslant M_\theta\|\eta-\theta\|^2. \quad (2.7)$$

It is evident that $\tilde{\eta}\in\hat{\Theta}$ is the "improved" estimation in comparison with the initial estimation $\eta\in X$.

The simplest unbiased estimate for the parameter $\theta\in\Theta$ is value ξ. Its projection on subspace $\hat{\Theta}$ let us designate $\tilde{\xi}$:

$$\tilde{\xi}=\sum_k\langle\xi,\theta_k\rangle\theta_k. \quad (2.8)$$

For the unbiased estimate $\tilde{\xi}$, called the estimation of least squares, we have

$$M_\theta\|\tilde{\xi}-\theta\|^2=\sum_k M_\theta(\langle\xi,\theta_k\rangle-\langle\theta,\theta_k\rangle)^2= =\sum_k\langle B^*\theta_k,\theta_k\rangle<\infty, \quad (2.9)$$

since the correlation operator B^* Gaussian value $\xi\in X$ it is nuclear (see §4 chapter I).

Each of real values $\eta_k = \langle \xi, \theta_k \rangle$, k = 1, 2, ..., n, is the unbiased estimate for a functional $\alpha_k(\theta) = \langle \theta, \theta_k \rangle$ from $\theta \in \Theta$. There are the best unbiased estimates $\hat{\alpha}_k$, $k = 1, 2, \ldots,$ described in theorem 1. For them

$$M_\theta(\hat{\alpha}_k - \langle \theta, \theta_k \rangle)^2 \leqslant M_\theta(\langle \xi, \theta_k \rangle - \langle \theta, \theta_k \rangle)^2, \quad k = 1, 2, \ldots \tag{2.10}$$

From relationship/ratios (2.9) and (2.10) it is evident that

$$M \sum_k \hat{\alpha}_k^2 = \sum_k M \hat{\alpha}_k^2 \leqslant \sum_k M \langle \xi, \theta_k \rangle^2 = \sum_k \langle B^* \theta_k, \theta_k \rangle < \infty,$$

and therefore with probability 1

$$\sum_k \hat{\alpha}_k^2 < \infty. \tag{2.11}$$

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Let us consider value $\hat{\theta} \in \Theta$, determined with probability 1 by the formula

$$\hat{\theta} = \sum_k \hat{\alpha}_k \theta_k, \tag{2.12}$$

where $\hat{\alpha}_k$ - best estimators for coefficients $\alpha_k(\theta) = \langle \theta, \theta_k \rangle$.

It is clear that $\hat{\theta}$ is the unbiased estimate for an unknown average value $\theta \in \Theta$, and, whatever the unbiased estimate $\eta \in X$, from inequality (1.27) we have

$$M_\theta \|\hat{\theta} - \theta\|^2 = \sum_k M_\theta (\hat{\alpha}_k - \langle\theta, \theta_k\rangle)^2 \leqslant$$
$$\leqslant \sum_k M_\theta (\langle\eta, \theta_k\rangle - \langle\theta, \theta_k\rangle)^2 \leqslant M_\theta \|\eta - \theta\|^2, \quad (2.13)$$

so that value $\hat{\theta}$ it is best (in the sense of the obtained inequality (2.13)) by the unbiased estimate for the parameter $\theta \in \Theta$.

Theorem 3. There is the best unbiased estimate $\hat{\theta}$ for the unknown parameter $\theta \in \Theta$, as any cell/element from function space X representable in the form

$$\hat{\theta}(t) = \int e^{-i\lambda t} \hat{\eta}(\lambda) G(d\lambda), \quad t \in T \quad (2.14)$$

(where $\hat{\eta}(\lambda) \in L_T$ (G)). With each fixed/recorded $t \in T$ the value $\hat{\theta}$ (t) gives the best unbiased estimate for a value θ (t)

unknown average $\theta \in \Theta$. Random process $\hat{\theta}$ (t), $t \in T$, is obtained by the linear transformation of the "observed" random process ξ (t), $t \in T$, determined by the formula

$$\hat{\theta}(t) = \mathscr{P}\xi(t), \quad t \in T, \quad (2.15)$$

where $\mathscr{P}$ - the operator of design in space H (T) to subspace L, generated by the values of form (1.25).

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Random process $\hat{\theta} = \hat{\theta}$ (t) allow/assumes expansion (see (1.31))

$$\hat{\theta}(t) = \int \varphi(\lambda, t) \Phi(d\lambda), \quad (2.16)$$

where $\phi(\lambda, t)$ is a projection of cell/element $e^{i\lambda t} \in L_T(F)$ on the subspace, generated by all functions $\varphi_0(\lambda)$, that figure in formula (1.16). Value $\hat{\eta}(\lambda) \in L_T(G)$ in representation (2.14) with probability 1 is

$$\hat{\eta}(\lambda) = \sum_k \hat{\alpha}_k \varphi_k(\lambda), \qquad (2.17)$$

where $\hat{\alpha}_k$, k = 1, 2, ..., the same random coefficients, as in initial formula (2.12), but the functions $\varphi_k(\lambda) \in L_T(G)$ correspond to cell/elements $\theta_k \in X$:

$$\theta_k(t) = \int e^{-i\lambda t}\varphi_k(\lambda)\, G(d\lambda), \quad t \in T, \quad k = 1, 2, \ldots \qquad (2.18)$$

Proof. According to (2.18)

$$\theta_k(t) = \langle e^{i\lambda t}, \varphi_k(\lambda)\rangle, \quad t \in T,$$

where ϕ_k, k = 1, 2, ..., orthonormal set in space $L_T(G)$, since θ_k, k = 1, 2, ..., orthonormal set in space X. Therefore

$$\sum_k \theta_k(t)^2 \leqslant \| e^{i\lambda t} \|_G^2 < \infty$$

with each fixed/recorded $t \in T$, and since $\sum_k \hat{\alpha}_k^2 < \infty$ with probability 1 (see (2.11)), the series $\hat{\theta}(t) = \sum_k \hat{\alpha}_k \theta(t), \quad t \in T, \qquad (2.19)$

with each fixed/recorded $t \in T$ descends also with probability 1. It is obvious, value $\hat{\theta}$ (t) (t fix/recorded) as limit at n ⟶ - values $\sum_{k=1}^{n} \hat{\alpha}_k \theta_k(t)$

from the locked subspace $L \subseteq H(T)$, belongs to L and is the best unbiased estimate for the unknown value

$$\theta(t) = \mathrm{M}_\theta \hat{\theta}(t) = \sum_k [\mathrm{M}_\theta \hat{\alpha}_k]\, \theta_k(t), \quad t \in T.$$

All the such estimations are described in theorem 1, from which escape/ensue formulas (2.15), (2.16). Theorem is demonstrated.

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Thus, there is best unbiased estimate $\hat{\theta} = \hat{\theta}$ (t), more precise than other unbiased estimates reproducing the unknown functional parameter $\theta = \theta$ (t) from subspace $\Theta \in X$.. This estimation $\hat{\theta} \in X$ enters in closing/shorting $\hat{\Theta}$ subspace Θ (see formula (2.12)). At the same time occurs the following result.

Theorem 4. The best unbiased estimate $\hat{\theta}$ belongs to the initial subspace Θ when and only when Θ is finite-dimensional.

Proof. By formula (2.16)

$$\hat{\theta}(t) = \int [\mathscr{P} e^{i\lambda t}] \Phi(d\lambda), \quad t \in T,$$

where $\mathscr{P}$ is an operator of design in hilbert space L_T (F) to subspace L, generated by all functions ϕ_0 (λ), connected with $\theta \in \Theta$ by equalities (1.16). This subspace L is unitary isomorphically Θ, as subspace in hilbert space Y (see (1.29)). If with the positive probability of the trajectory of certain equivalent random process $\hat{\theta} =$

= $\hat{\theta}$ (t), t $\in$ T, they enter in Y, then there exists Gaussian function $\zeta(\lambda) \in L_T(F)$ such, that with probability 1

$$\hat{\theta}(t) = \langle e^{i\lambda t}, \zeta(\lambda)\rangle_F, \quad t \in T$$

(see §6 chapter I), correlation operator of which can be determined from the relationship/ratios

$$\mathsf{M}[\langle e^{i\lambda s}, \zeta(\lambda)\rangle_F \langle e^{i\lambda t}, \zeta(\lambda)\rangle_F] = \mathsf{M}[\hat{\theta}(s)\hat{\theta}(t)] = \\ = \langle \mathscr{P}e^{i\lambda s}, \mathscr{P}e^{i\lambda t}\rangle_F = \langle \mathscr{P}^*\mathscr{P}e^{i\lambda s}, e^{i\lambda t}\rangle_F, \quad s, t \in T.$$

It is evident that this correlation operator is $\mathscr{P}^*\mathscr{P} = \mathscr{P}$. Correlation operator must be nuclear (see theorem 1 chapter I), and the operator of design $\mathscr{P}$ possesses this property if and only if the subspace L (and, consequently, unitary isomorphic to him space Θ) is finite-dimensional. Theorem is demonstrated.

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Let us pause at the finite-dimensional case. The best unbiased estimate $\hat{\theta} = \hat{\theta}$ (t) for unknown average value $\theta - \theta$ (t), t $\in$ T, it can be represented in the form

$$\hat{\theta} = \sum_{k=1}^{N} \eta_k \theta_k,$$

where $\eta_k = \hat{\alpha}_k$ – the best unbiased estimates of coefficients α_k = <θ, θ_k>, $\theta \in \Theta$, in the expansion

$$\theta = \sum_{k=1}^{N} \alpha_k \theta_k$$

cell/element $\theta \in \Theta$ in terms of base cell/elements $\theta_1, \ldots, \theta_N$. In the case of the orthonormalized base $\theta, \ldots, \theta_N$ the corresponding estimations $\eta_1, \ldots, \eta_N$ are independent Gaussian values with identical (equal to 1) dispersion (see theorem 2).

The best unbiased estimate $\hat{\theta}$ is the point of the maximum of the "function of the plausibility" of $l(\theta) = \log p_\theta$ of $\theta \in \Theta$, where $p_\theta = p_\theta(\omega)$ there is density $P_\theta(d\omega)/P(d\omega)$, determined from formula (1.15):

$$l(\theta) = \eta_\theta - \frac{1}{2}\|\eta\|^2 = \sum_k a_k \eta_k - \frac{1}{2}\sum_k a_k^2.$$

Actually, it is evident that the maximum of function $l(\theta)$ is reached when $a_k = \eta_k$ $(k = 1, \ldots, N)$:

$$\max_{\theta \in \Theta} l(\theta) = \frac{1}{2}\sum_k \eta_k^2 = l(\hat{\theta}).$$

It should be noted that the "method of the maximum of plausibility" is inapplicable in infinite-dimensional case, when the Gaussian process $l(\theta)$, $\theta \in \Theta$, is not limited.

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This is immediately evident, for example, from the relationship/ratio: with probability 1

$$\sup_{\theta \in \Theta} l(\theta) \geqslant \sup(\eta_1, \eta_2, \ldots) - \frac{1}{2} = \infty,$$

where η_k, k = 1, 2, ..., the infinite sequence of independent Gaussian values with equal to 1 dispersion and the average values $\alpha_k = \langle\theta, \theta_k\rangle$, such that $\sum_k \alpha_k^2 < \infty$; here $\theta_1, \theta_2, \ldots$, the orthonormalized base in infinite-dimensional subspace $\Theta \subseteq Y$, and η_1, η_2, ..., the corresponding values in subspace $L \subseteq H(T)$ (see theorem 2).

But one should also say that the formally written conditions of extremum for the function of plausibility $l(\theta)$ of $\theta = \sum_k \alpha_k \theta_k$:

$$\frac{\partial l(\theta)}{\partial \alpha_k} = 0, \quad k = 1, 2, \ldots,$$

lead to the best unbiased estimates $\eta_k = \hat{\alpha}_k$ for the appropriate coefficients $\alpha_k = \langle\theta, \theta_k\rangle$, k = 1, 2, ..., but, since "coefficients" $\hat{\alpha}_k = \eta_k$, k = 1, 2, ..., are such, that from probability 1 series $\sum \eta_k^2$ diverges, in hilbert space Y there is no cell/element, representable by a series $\sum \hat{\alpha}_k \theta_k$. True, the best unbiased estimate $\hat{\theta}(t)$ with each fixed/recorded $t \in T$ is represented by converging series (2.19):

$$\hat{\theta}(t) = \sum_k \eta_k \theta_k(t).$$

Above we have used hilbert space $L_m(G)$, corresponding to the final spectral measure $G(d\lambda)$, what made it possible to be converted directly to spectral representation (2.2) for the separate values of

the initial random process ξ (t), $t \in T$, coinciding during functions $\phi(\lambda) = e^{i\lambda t}$ with the values of the random functional

$$\eta(\varphi) = \langle \varphi, \eta \rangle_G, \quad \varphi \in L_T(G),$$

which relative to distribution P_0 it has the average value $\psi_0(\lambda) \in L_T(G)$, where ψ_0 it assigns linear functional in hilbert space $L_T(G)$, that coincides during functions $\phi(\lambda) = e^{i\lambda t}$ with values θ (t), $t \in T$. (see (2.3), (2.4)).

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From the very beginning instead of the random process ξ (t) of the continuous parameter $t \in T$ it would be possible to examine the generalized random process $\eta(\phi)$ of $\varphi \in L_T(G)$ with the average value ψ_0:

$$\langle \varphi, \psi_0 \rangle_G = M_0 \langle \varphi, \eta \rangle_G, \quad \varphi \in L_T(G),$$

and by the correlation functional B = A*A, where operator A from $L_T(G)$ in $L_T(F)$ functions by formula $A\phi = \phi$. Here it is appropriate to recall (see §6 chapter I) that space itself $L_T(g)$ can be defined as closing/shorting of the space of all functions of the form

$$\varphi(\lambda) = \int_T e^{i\lambda t} c(t)\, dt,$$

where c (t) - the infinitely differentiated function, which turns in 0 outside interval of T.

In the examination of the random functional of form (2.2) or, which is the same thing, random element η in hilbert space L_T (G), the speech must go about the evaluations of the average value $\psi_0 \in L_T$ (G). All the developed above methods and results are used also to the described diagram; necessary only of the parametric multitude T to pass to the parametric set L_T (G) and of values ξ (t) and θ (t), t $\in$ T, to values $\langle \phi, \eta \rangle_\xi$ and $\langle \phi, \psi_\theta \rangle_\theta$, $\phi \in L_T$ (G).

A special case is G (dλ) = 1/2π dλ. Here L_T (G) coincides with the function space of the form

$$\varphi(\lambda) = \int_T e^{i\lambda t} c(t)\, dt,$$

where c (t) $\in \mathscr{L}^2(T)$ - to the space of all integrated squared functions, whereupon on the strength of the equality of Parseval

$$\langle \varphi_1, \varphi_2 \rangle_G = \langle c_1(t), c_2(t) \rangle = \int c_1(t)\, c_2(t)\, dt.$$

It is clear that to examine random function $\eta(\lambda) \in L_T$ (G), that has the form

$$\eta(\lambda) = \int_T e^{i\lambda t} \xi(t)\, dt,$$

or directly initial random process ξ (t), t $\in$ T as function in hilbert space $\mathscr{L}^2$ (T) are actually one and the same (let us note that $\psi_0(\lambda) = \int_T e^{i\lambda t} \theta(t)\, dt$).

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§3. Best estimators and their justifiability.

Let us assume that the "observed" random process $\xi(t)$, $t \in T$, is cell/element of that which was determined previously hilbert space X of the real functions x = x (t) of $t \in T$ of form (2.5). Let us consider value $\theta \in X$ - the estimation of least squares for an unknown average value $\theta \in \Theta, \Theta \subseteq X$, which is given by formula (2.8).

In expression (2.9) figure the estimations

$$\tilde{\alpha}_k = \langle \theta_k, \xi \rangle = \langle \theta_k, \theta \rangle$$

for coefficients $\alpha_k = \langle \theta, \theta_k \rangle$, k = 1, 2, ..., in the expansion of cell/element $\theta \in \Theta$ in terms of the orthonormalized base $\theta_1, \theta_2, \ldots \in \tilde{\Theta}$. These estimations and generally the evaluations of the form

$$\tilde{\alpha} = \langle \theta_\alpha, \theta \rangle = \langle \theta_\alpha, \xi \rangle$$

for functionals

$$\alpha(\theta) = \langle \theta_\alpha, \theta \rangle \text{ or } \theta \in \Theta \qquad (3.1)$$

in hilbert space X (where $\theta_\alpha \in \tilde{\Theta}$) let us call the estimations of least squares.

The estimations of least squares can be represented in the following spectral form:

$$\tilde{\alpha} = \int \varphi_\alpha(\lambda)\, \Phi(d\lambda), \qquad (3.2)$$

where φ_α **(λ)** $\in$ **L_T (G) is a solution to the integral** equation

$$\theta_\alpha(t) = \int e^{-i\lambda t}\, \varphi_\alpha(\lambda)\, G(d\lambda), \qquad t \in T. \qquad (3.3)$$

In fact, from results §6 chapter I follows that

$$\tilde{\alpha} = \langle \theta_\alpha, \theta \rangle = \langle \theta_\alpha, \xi \rangle = \langle \varphi_\alpha, \eta \rangle = \int \varphi_\alpha(\lambda)\, \Phi(d\lambda),$$

where η **(λ)** $\in L_T(G)$ **- random function from spectral represen**tation (2.2).

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Let us note that in the case of G (dλ) = (1/2π) dλ we deal with the classical estimations of the least squares, when "observed" random **process** $\xi = \xi$ **(t), t** $\in$ **T, is considered** as point **in the** classical **hilbert space** $\mathcal{L}^2(T)$ **real functions** x = x **(t) of** t $\in$ **T with** the scalar **product**

$$\langle x_1, x_2 \rangle = \langle \psi_1, \psi_2 \rangle_G = \begin{cases} \sum\limits_T x_1(t)\, x_2(t) \text{ при дискретном } t, (1) \\ \int\limits_T x_1(t)\, x_2(t)\, dt \text{ при непрерывном } t, (2) \end{cases}$$

Key: (1). with discrete t. (2). with continuous t.

where $\psi_1, \psi_2 \in L_T(G)$ are connected with x_1, $x_2 \in \mathcal{L}^2(T)$ by usual Fourier transform

$$\psi(\lambda) = \begin{cases} \sum_t e^{i\lambda t} x(t) & \text{при дискретном } t, \ (1) \\ \int_T e^{i\lambda t} x(t)\, dt & \text{при непрерывном } t. \ (2) \end{cases}$$

Key: (1). with discrete t. (2). with continuous t.

The very evaluations of type (3.3) for functionals $\alpha(\theta) = \langle \theta_\alpha, \theta \rangle$ from $\theta \in \Theta \subseteq \mathcal{L}^2(T)$ has the form

$$\tilde{\alpha} = \begin{cases} \sum_t \theta_\alpha(t)\xi(t) & \text{при дискретном } t, \ (1) \\ \int_T \theta_\alpha(t)\xi(t)\, dt & \text{при непрерывном } t. (2) \end{cases} \quad (3.4)$$

Key: (1). with discrete t. (2). with continuous t.

Formula (3.2) in its structure coincides with formula (1..31) (see theorem 1), so that if G (dλ) was the spectral measure of stationary process ξ (t), t ∈ T, then estimation $\tilde{\alpha}$ form (3.2) they would be the best unbiased estimates for functionals of type (3.1).

Below we will consider the new class of estimations, which

generalizes the estimations of least squares and which includes as extreme case and best unbiased estimates.

Specifically, formulas (3.2), (3.3) make sense not only when the trajectory of the "observed" process $\xi = \xi(t)$, $t \in T$, enter in the appropriate hilbert space X, but even when are only satisfied conditions (indicated in the beginning §2)

$$L_T(G) \subseteq L_T(F), \quad \|\varphi\|_F \leqslant \|\varphi\|_G, \quad \varphi \in L_T(G). \qquad (3.5)$$

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If G ($d\lambda$) was the spectral measure of stationary process $\xi = \xi(t)$, then formulas (3.2), (3.3) would give the best unbiased estimates for functionals of type (3.1). Let us name measure G ($d\lambda$), with the help of which are constructed estimations (3.2) a pseudo spectral measure, and the estimations themselves (3.2) -pseudo best estimators (for functionals of the type (3.1)).

It occurs the following of the formulas:

$$M_\theta \int \varphi(\lambda)\Phi(d\lambda) = \langle \varphi, \psi_\theta \rangle_G, \quad \varphi \in L_T(G), \qquad (3.6)$$

where $\psi_\theta(\lambda) \in L_T(G)$ - function in spectral representation (2.1) for $\theta \in \Theta$.

In the case, when random process $\xi = \xi(t)$, $t \in T$, is random element in Hilbert space X (constructed according to the

pseudospectral measure G (dλ)), this formula was already establish/installed (see (2.3)). In the case of the arbitrary final measure G(dλ) with ϕ (λ) = $e^{i\lambda t}$ formula (3.6) coincides with (2.1) and extends to closed linear shell of functions ϕ (λ) = $e^{i\lambda t}$ (t ∈ T), i.e., to whole hilbert space L_T (G) (comp. (1.22)). .

According to (3.6)

$$M_\theta\tilde{\alpha} = M_\theta \int \varphi_\alpha(\lambda)\,\Phi(d\lambda) = \langle\varphi_\alpha, \psi_\theta\rangle_G = \langle\theta_\alpha, \theta\rangle = \alpha(\theta), \quad \theta \in \Theta, \tag{3.7}$$

and thus the best estimators are unbiased.

The important property of pseudo best estimators is the invariance relative to the multiplication of pseudospectral measure $G(d\lambda)$ by any constant factor k.

Actually, the function ϕ_α (λ) in formula (3.2) unambiguously is determined from the relationship/ratio

$$\alpha(\theta) = \langle\varphi_\alpha, \psi_\theta\rangle_G, \quad \theta \in \Theta, \tag{3.8}$$

where the function $\psi_\theta(\lambda) \in L_T(G)$ and θ (t) ∈ Θ are connected by formula (2.1).

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It is obvious,

$$\theta(t) = \int e^{-i\lambda t}\,[k^{-1}\psi_\theta(\lambda)]\,kG(d\lambda)$$

and

$$\langle\varphi_\alpha, k^{-1}\psi_\theta\rangle_{kG} = \langle\varphi_\alpha, \psi_\theta\rangle_G = \alpha(\theta),$$

so that with respect to the pseudospectral measure kG (dλ) function $\varphi_n(\lambda)$ in formula (3.2) is the same as with respect to the initial **measure** G (dλ).

Let us assume that we deal with the sequence of the "being widened observations", considering random process $\xi = \xi$ (t) on sets T = T_1, T_2, ...;

$$T_1 \subseteq T_2 \subseteq \ldots$$

The question will be **about best** estimators of $\tilde{\alpha}_n$ type (3.2), constructed according to the "observations" of random process $\xi = \xi$ (t) on the appropriate multitude T = T_n.

Let α (θ) - linear functional of unknown average value $\theta = \theta$ (t), t $\in$ T *, where

$$T^* = \bigcup T_n,$$

continuous with certain n = n_0 relative to the scalar product

$$\langle\theta_1, \theta_2\rangle_n = \langle\psi^n_{\theta_1}, \psi^n_{\theta_2}\rangle_G \qquad (3.9)$$

of the same type as (2.6), namely function $\theta = \theta$ (t), t $\in$ T *, from parametric space Θ and function $\psi^n_\theta(\lambda) \in L_{T_n}(\dot{G})$ are connected by

relationship of type (2.1):

$$\theta(t)=\int e^{-i\lambda t}\psi_0^n(\lambda)\,G(d\lambda),\quad t\in T_n. \qquad (3.10)$$

Lemma 5. The scalar products indicated are such, that

$$\|\theta\|_1\leqslant\|\theta\|_2\leqslant\ \dots \qquad (3.11)$$

Proof. In fact, if representation (3.10) occurs with certain T_n, then, after taking projection ψ_0^m cell/element $\psi_0^n\in L_{T_n}(G)$ on the subspace

$$L_{T_m}(G)\subseteq L_{T_n}(G),\ m\leqslant n,\ \text{будем иметь также}\ (1)$$
$$\theta(t)=\langle e^{i\lambda t},\ \psi_0^n\rangle_G=\langle e^{i\lambda t},\ \psi_0^m\rangle_G,\quad t\in T_m,$$

Key: (1). let us have also

i.e. representation (3.10) it occurs, also, with every ~~one~~ $T_m\subseteq T_n$, whereupon corresponding cell/element ψ_0^m is the projection on $L_{T_m}(G)$ of parent element $\psi_0^n\in L_{T_n}(G)$.

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Hence it follows that

$$\|\theta\|_m=\|\psi_0^m\|_G\leqslant\|\psi_0^n\|_G=\|\theta\|_n$$

with any $m\leqslant n$, Q. E. D.

Inequalities (3.11) make it possible to conclude that if the

linear functional α (θ) of $\theta \in \Theta$ **with certain** $n = n_0$ is continuous relative to scalar product **(3.9), then it will possess this same** property as ~~with all $n \geqslant n_0$,~~ (so that for any $n \geqslant n_0$,) **takes place a** representation of type **(3.1), (3.8):**

$$\alpha(\theta) = \langle \varphi_\alpha^n, \psi_\theta^n \rangle_G, \quad \theta \in \Theta, \qquad (3.12)$$

which it **answers the pseudobest unbiased estimate of** type (3.2):

$$\tilde{\alpha}_n = \int \varphi_\alpha^n(\lambda)\, \Phi(d\lambda). \qquad (3.13)$$

Let us name **estimations** $\tilde{\alpha}_n,\ n \geqslant n_0,$ **justified, if**

$$M_\theta[\tilde{\alpha}_n - \alpha(\theta)]^2 = \|\varphi_\alpha^n\|_F^2 \to 0 \qquad (3.14)$$

with $n \longrightarrow \infty$.

Let us assume temporarily that G $(d\lambda)$ is true spectral measure and, in accordance with this, $\tilde{\alpha}_n$ - **the** best **unbiased** estimates, constructed according to "**observations**" ξ (t), t $\in T_n$.

Recall (see theorem 1) that in this case **the best** estimator $\tilde{\alpha}$, constructed according to "**observations**" ξ (t), t $\in T$, **is** the only **unbiased** estimate in space **L** $\subseteq$ **H (T)**, generated **by all values** $\eta_\theta,\ \theta \in \Theta$, **form** (1.25). In the terms **appropriate** hilbert space L_T (G) **(G** $(d\lambda)$ - spectral measure) this means that among all **functions** ϕ $(\lambda) \in L_T$ (G), that assign the unbiased **estimates** $\int \varphi(\lambda)\Phi(d\lambda)$ **for the** functional α ~~in question~~ (θ) **of** $\theta \in \Theta$, only **assigning best** estimator

the function φ_α (λ) lie/rests at subspace $L \subseteq L_T$ (G), which is generated by all possible cell/elements $\psi_0(\lambda) \in L_T$ (G) from (2.1); in this case φ_α (λ) coincides with projection on the subspace L indicated $L \subseteq L_T(G)$ "spectral characteristic" ϕ (λ) any other unbiased estimate $\int \varphi(\lambda) \Phi(d\lambda)$.

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Accordingly $\varphi_\alpha^n(\lambda)$ coincides with projection on $L_n \subseteq L_{T_n}(G)$ any of the functions $\varphi_\alpha^m(\lambda)$ (with m ≤ n), where L_n indicates the subspace, generated by all functions $\psi_0^n(\lambda) \in L_{T_n}(G)$, $\theta \in \Theta$.

Consequently,

$$\|\varphi_\alpha^m - \varphi_\alpha^n\|_G^2 = \|\varphi_\alpha^m\|_G^2 - \|\varphi_\alpha^n\|_G^2 \to 0$$

with m, n —> ∞ exists the limit

$$\varphi_\alpha^*(\lambda) = \lim_{n\to\infty} \varphi_\alpha^n(\lambda) \qquad (3.15)$$

(in space L_{T^*}(G)). By passing to space L_{T^*}(F) (under condition (3..5) for T = T *), we come to the existence of cell/element $\varphi_\alpha^*(\lambda) \in L_T(F)$ such, that

$$\|\varphi_\alpha^n - \varphi_\alpha^*\|_F \to 0. \qquad (3.16)$$

It is evident (see (3.14)), that justifiability occurs when and only when

$$\varphi_\alpha^*(\lambda) = 0 \qquad (3.17)$$

(almost everywhere relative to F (dλ)).

Thus, under further assumption about the fact that measure G (dλ) is equivalent to the true spectral measure F (dλ) and (comp. (3.5))

$$L_{T^*}(G) \subseteq L_{T^*}(F), \qquad (3.18)$$
$$\|\varphi\|_F \leqslant \|\varphi\|_G, \quad \varphi \in L_{T^*}(G),$$

for a consistency of estimator $\tilde{\alpha}_n$ necessary and it is sufficient in order that limit function $\varphi^*_\alpha(\lambda)$ in (3.15) would be equal to 0 almost everywhere relative to G (dλ); it is obvious, this is equivalent also to the condition

$$\lim_{n\to\infty} \|\varphi^n_\alpha\|_G = 0. \qquad (3.19)$$

Let us define $\tilde{\Theta}_n$ as space of all real functions θ = θ (t), t $\in T^*$, which is the closing/shorting of parametric space Θ relative to norm $\|\theta\| = \|\psi^n_\theta\|_G$, and assume

$$\tilde{\Theta} = \bigcap \tilde{\Theta}_n.$$

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Let us consider function

$$\theta^*(t) = \int e^{-i\lambda t} \varphi^*_\alpha(\lambda)\, G(d\lambda), \quad t \in T^*, \qquad (3.20)$$

connected by a formula of type (3.10) with limit function $\varphi^*_\alpha(\lambda)$. It is easy to see that $\theta^* \in \tilde{\Theta}$, since θ * (t) there is a limit (in each

fixed/recorded space $\tilde{\Theta}_m$) of the sequence of the functions

$$\theta^n(t) = \int e^{-i\lambda t}\varphi_\alpha^n(\lambda)\, G(d\lambda), \quad n \geqslant m.$$

Any linear functional $\alpha(\theta)$ from $\theta \in \Theta$ of the considered type (i.e., continuous in some norm $\|\theta\|_m = \|\psi_\theta^m\|_G$) with respect to continuity is spread to closure $\tilde{\Theta}_m$ and thereby to space $\tilde{\Theta}$. Specifically, from formula (3.12) in question from continuity is determined and the value

$$\alpha(\theta^*) = \lim_{n\to\infty} \alpha(\theta^n) = \lim_{n\to\infty} \langle \varphi_\alpha^n, \varphi_\alpha^n \rangle_G = \|\varphi_\alpha^*\|_G^2.$$

Thus, for the justifiability of best estimators $\tilde{\alpha}_n$ it is necessary and sufficiently, in order to

$$\alpha(\theta^*) = 0. \qquad (3.21)$$

Without limiting generality, it is possible to count that $\Theta = \tilde{\Theta}$, since the class of the pseudobest unbiased estimates for an initial parametric multitude Θ coincides with the class of the pseudobest unbiased estimates for its locked (in above sense indicated) linear closure $\tilde{\Theta}$ (see in regard to this observation on page 324). In connection with this let us assume that

$$\tilde{\Theta} = \Theta.$$

Let us designate Θ^* subspace in Θ, formed by all functions $\theta = \theta(t)$, $t \in T^*$, which allow/assume a spectral representation of type (3.10):

$$\theta(t) = \int e^{-i\lambda t}\psi_\theta(\lambda)\, G(d\lambda), \quad t \in T^*. \qquad (3.22)$$

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For cell/elements $\theta \in \Theta^{\bullet}$ from formulas (3.10), (3.12) (valid with all $n \geqslant n_0$) we obtain

$$\alpha(\theta) = \langle \varphi_\alpha^n, \psi_\theta \rangle_G = \langle \varphi_\alpha^{\bullet}, \psi_\theta \rangle_G, \tag{3.23}$$

whence in the case of the justifiability of best estimators $\tilde{\alpha}_n$ it follows that

$$\alpha(\theta) = 0 \text{ with } \theta \in \Theta^{\bullet}. \tag{3.24}$$

Together on condition of justifiability (3.21) this gives the following result.

Theorem 5. For the justifiability of best estimators $\tilde{\alpha}_n$ linear functional α (θ) from $\theta \in \Theta$ it is necessary and sufficiently condition (3.24).

Let us note that if subspace $\Theta^{\bullet}$ all functions $\theta = \theta$ (t), $t \in T$, which allow/assume spectral representation (3.22), contains only trivial cell/element θ (t) $\equiv 0$, then for any linear functional α (θ), $\theta \in \Theta$, carried out condition (3.24) and for theorem 5 all the best estimators (for all linear functionals α (θ) of $\theta \in \Theta$) will be

justified. Conversely, the justifiability of all best estimators means that (see (3.24)

$$\alpha(\theta) = \langle \varphi_a^n, \psi_0 \rangle_G = 0$$

for all linear functionals (for all cell/elements $\varphi_a^n \in L_{T_n}$ (G), n = 1, 2, ...) and, therefore, ψ_0 = 0, i.e., subspace Θ^* trivially.

Let us find the "spectral conditions", necessary and sufficient in order that

$$\Theta^* = 0, \qquad (3.25)$$

i.e. in order that not one of functions θ (t) $\in \Theta$ would allow/assume representation (3.22).

Let us assume that the pseudospectral measure G (dλ) is absolutely continuous, and spectral density g (λ) = G (dλ) /dλ is limited.

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If under these conditions function θ (t), t $\in T^*$, is represented by formula (3.22), then it with t $\in T$ coincides with the summarized squared function θ (t),-∞ < t < ∞, the being Fourier transform integrated squared function $\tilde{\theta}$ (λ) = ψ_0 (λ) g (λ), by satisfying to the same condition

$$\int \frac{|\tilde{\theta}(\lambda)|^2}{g(\lambda)} d\lambda < \infty \qquad (3.26)$$

(comp. on condition (4.6) chapter III). On the other hand, if there is the described type function θ (t),$-\infty < t < \infty$, then with $t \in T^*$ it allow/assumes representation (3.22), in which ψ_θ (λ) $\in L_{T^*}$ (G) is a projection of the function $\theta(\lambda)/g(\lambda)$ from $L_{(-\infty,\infty)}(G)$ for subspace $L_{T^*}(G)$.

Thus, in the case of the limited spectral density g (λ) we arrived at the following result.

Theorem 6. For the justifiability of best estimators it is necessary and sufficiently in order that each function θ (t), $t \in T^*$, from the locked parametric space Θ would possess those by property, that either θ (t), $t \in T^*$, is not summarized squared [1], or for any continuation before the summarized squared function θ (t), $-\infty < t < \infty$, Fourier's transform $\tilde{\theta}$ (λ) it satisfies the condition

$$\int \frac{|\tilde{\theta}(\lambda)|^2}{g(\lambda)} d\lambda = \infty. \qquad (3.27)$$

FOOTNOTE [1]. I. e.,

$$ ^{1)}\ \sum_{-\infty}^{\infty} \theta(t)^2 = \infty \text{ при дискретном } t \text{ и } (1) $$

$$ \int_{-\infty}^{\infty} \theta(t)^2\,dt = \infty \text{ при непрерывном } t. \quad (2) $$

Key: (1). with discrete t and. (2). with continuous t. ENDFOOTNOTE.

Simple corollary of this is following necessary ii a sufficient condition of the justifiability of the classical estimations of least squares (corresponding to spectral density g (λ) = 1):

$$ \begin{aligned} &\sum_{t \in T^*} \theta(t)^2 = \infty \text{ при дискретном } t, \ (1) \\ &\int_{T^*} \theta(t)^2\,dt = \infty \text{ при непрерывном } t. (2) \end{aligned} \quad (3.28) $$

Key: (1). with discrete t. (2). with continuous t.

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Generally speaking, the subspace Θ' all functions of type (3.22) is significant.

Theorem 7. For a parametric multitude Θ^*, best estimator $\tilde{\alpha}_n$ descend with n —> ∞ to best estimator $\tilde{\alpha}$ (the same functional α (θ), $\theta \in \Theta^*$), constructed according to observations ξ (t), $t \in T^*$.

Proof. Relationship/ratio (3.23) shows that

$$\alpha(\theta) = \langle \varphi_\alpha^*, \psi_\theta \rangle_Q, \quad \theta \in \Theta^*, \tag{3.29}$$

and together with the asymptotic relation

$$\tilde{\alpha}_n = \int \varphi_\alpha^n \Phi(d\lambda) \to \int \varphi_\alpha^* \Phi(d\lambda)$$

this proves theorem, since according to common/general/total formula (3.2) the best estimator for the functional of form (3.29) is

$$\alpha^* = \int \varphi_\alpha^*(\lambda) \Phi(d\lambda).$$

§4. Estimations of regression coefficients.

1. Some generalities. Let us pause at the case, when the parametric space Θ is finite-dimensional and the question concerning the estimations of linear functionals α (θ) of $\theta \in \Theta$ examine/considered by us type (3.1) is reduced to the question concerning the estimations of coefficients α_1, ..., α_N in the expansion of function θ (t) from Θ in terms of certain base θ_1 (t),

..., θ_N (t):

$$\theta(t) = \alpha_1\theta_1(t) + \ldots + \alpha_N\theta_N(t), \quad t \in T^{*}. \qquad (4.1)$$

Let us isolate subspace Θ^{*} all functions $\theta = \theta(t), t \in T^{*}$, from Θ, that allow/assume spectral representation (3.22). Let us select in subspace $\Theta^{*} \subseteq \Theta$ certain base $\theta_1, \ldots, \theta_N$, in all space Θ.

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Let $T_1 \subseteq T_2 \subseteq \ldots$ - the sequence of the sets $\left(\bigcup_n T_n = T^{*}\right)$, for each of which all the functions θ_k (t), k = 1, ..., N, allow/assume spectral representation (3.10):

$$\theta_k(t) = \int e^{-i\lambda t}\psi_k^n(\lambda)\, G(d\lambda), \quad t \in T_n. \qquad (4.2)$$

Let $\tilde{\alpha}_1^n, \ldots, \tilde{\alpha}_N^n$ — best estimators for coefficients $\alpha_1, \ldots, \alpha_N$:

$$\tilde{\alpha}_k^n = \int \varphi_k^n(\lambda)\, \Phi(d\lambda) \qquad (4.3)$$

(k = 1, ..., N); in this formula $\phi_1^n, \ldots, \phi_N^n$ - the interconnected circuit of cell/elements to $\psi_1^n, \ldots, \psi_N^n$ (see formula (1.40)), namely:

$$\varphi_k^n(\lambda) = \sum_{j=1}^{N} s_{kj}^n \psi_j^n(\lambda), \qquad (4.4)$$

where the matrix/die $\{s_{kj}^n\}$ is reverse to matrix/die with the cell/elements

$$\langle\psi_k^n, \psi_j^n\rangle_G = \int \psi_k^n(\lambda)\overline{\psi_j^n(\lambda)}\, G(d\lambda),$$
$$k, j = 1, \ldots, N.$$

whereupon

$$s^n_{kj} = \langle \varphi^n_k, \varphi^n_j \rangle_G, \quad k, j = 1, \ldots, N.$$

From formulas (4.2) and (4.4) we have

$$\int e^{-i\lambda t} \varphi^n_k(\lambda)\, G(d\lambda) = \sum_{j=1}^{N} s^n_{kj} \int e^{-i\lambda t} \psi^n_j(\lambda)\, G(d\lambda) = \sum_{j=1}^{N} s^n_{kj} \theta_j(t), \quad t \in T_n.$$

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Passing here to limit with n ⟶ ∞, we obtain, that

$$\sum_{j=1}^{N} s^*_{kj} \theta_j(t) = \int e^{-i\lambda t} \varphi^*_k(\lambda)\, G(d\lambda), \quad t \in T^*,$$

where

$$\varphi^*_k(\lambda) = \lim_{n\to\infty} \varphi^n_k(\lambda) \in L_{T^*}(G)$$

and

$$s^*_{kj} = \lim_{n\to\infty} s^n_{kj} = \lim_{n\to\infty} \langle \varphi^n_k, \varphi^n_j \rangle_G$$

(see asymptotic relation (3.15). It is evident that linear combinations $\sum_{j=1}^{N} s^*_{kj}\theta_j(t)$ base cell/elements $\theta_1, \ldots, \theta_N$ they enter in subspace Θ^* with base cell/elements $\theta_1, \ldots, \theta_{N^*}$, which can be only in the case, when

$$\sum_{j=N^*+1}^{N} s^*_{kj}\theta_j(t)=0, \quad k=1, \ldots, N.$$

But since $\theta_{N^*+1}, \ldots, \theta_N$ are linearly independent, obtained equations are possible only under the condition, when

$$s^*_{kk}=\|\varphi^*_k(\lambda)\|^2_G=0, \quad k=N^*+1, \ldots, N.$$

These equalities mean that

$$\lim_{n\to\infty}\tilde{\alpha}^n_k=\alpha_k, \quad k=N^*+1, \ldots, N, \qquad (4.5)$$

i.e.

the method of the pseudobest estimations gives error-free estimation for coefficients α_k, k = N * + 1, ..., N. For remaining coefficients $\alpha_1, \ldots, \alpha_{N^*}$ best estimators $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{N^*}$ according to observations $\xi(t)$, $t \in T^*$, are described above formulas indicated, namely:

$$\tilde{\alpha}_k=\int \varphi_k(\lambda)\Phi(d\lambda), \quad k=1, \ldots, N^*,$$

where

$$\varphi_k(\lambda)=\sum_{j=1}^{N^*} s_{kj}\psi_j(\lambda),$$

the function $\psi_k(\lambda)$ are determined from the equations

$$\theta_k(t) = \int e^{-i\lambda t} \psi_k(\lambda)\, G(d\lambda), \quad t \in T^* \qquad (4.6)$$

$$(k = 1, \ldots, N^*), \quad u$$

$$\{s_{kj}\} = \{\langle \varphi_k, \varphi_j \rangle_G\} = \{\langle \psi_k, \psi_j \rangle_G\}^{-1}. \qquad (4.7)$$

Let us assume that all the estimations $\tilde{a}_1^n, \ldots, \tilde{a}_N^n$ are justified.

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From the results previous §3 it is easy to deduce, which for this is necessary and is sufficient the condition: with any real $c_1, \ldots, c_N$

$$\lim_{n \to \infty} \left\| \sum_{k=1}^{N} c_k \psi_k^n(\lambda) \right\|_G = \infty.$$

Let us visualize that certain formula

$$H_{kj}^n(d\lambda) = \frac{\psi_k^n(\lambda)\, \overline{\psi_j^n(\lambda)}}{\|\psi_k^n\|_G \, \|\psi_j^n\|_G} G(d\lambda) \qquad (4.8)$$

(composite) measure $H_{kj}^n(d\lambda)$ with $n \longrightarrow \infty$ weakly converge to certain measure $H_{kj}(d\lambda)$, i.e., for any limited and continuous function $\varphi(\lambda)$

$$\lim_{n \to \infty} \int \varphi(\lambda) H_{kj}^n\, d\lambda = \int \varphi(\lambda) H_{kj}(d\lambda).$$

Let us assume that the weak convergence $H_{kl}^n \Rightarrow H_{kl}$ occurs with all k, j = 1, ..., N, as the otyuer words, occurs the weak convergence of the matrix measures $H^n \Longrightarrow H$:

$$H^n = \{H_{kl}^n\}, \quad H = \{H_{kl}\}.$$

Let us name function ϕ (λ) H - permissible, if under the condition of the weak convergence $H^n \Longrightarrow H$ occurs the relationship/ratio

$$\lim_{n \to \infty} \int \varphi(\lambda) H^n(d\lambda) = \int \varphi(\lambda) H(d\lambda). \tag{4.9}$$

The permissible functions form the linear class, locked relative to uniform convergence and containing not only the limited continuous, but also sectionally continuous functions with the finite gap count in points λ the zero measure H (λ) = 0 (and, of course, the evenly locked linear closure of such functions).

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If we introduce the diagonal matrix/die

$$D_n = \begin{pmatrix} \|\psi_1^n\|_a & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \|\psi_N^n\|_a \end{pmatrix}, \tag{4.10}$$

that for a correlation matrix/die $\{\sigma_{kl}^n\}$ estimations $\tilde{a}_1^n, \ldots, \tilde{a}_N^n$

from formulas (4.3), (4.4) it is possible to obtain the following expression:

$$\{\sigma_{kj}^n\} = \left\{ \int \varphi_k^n(\lambda)\overline{\varphi_j^n(\lambda)}\, F(d\lambda) \right\} =$$
$$= \{s_{kj}^n\} \left\{ \int \psi_k^n(\lambda)\overline{\psi_j^n(\lambda)}\, F(d\lambda) \right\} \{s_{kj}^n\} =$$
$$= \{s_{kj}^n\} D_n \left\{ \int h(\lambda) \frac{\psi_k^n(\lambda)\overline{\psi_j^n(\lambda)}}{\|\psi_k^n\|_G \|\psi_j^n\|_G} G(d\lambda) \right\} D_n \{s_{kj}^n\} =$$
$$= \{s_{kj}^n\} D_n \left\{ \int h(\lambda) H_{kj}^n(d\lambda) \right\} D_n \{s_{kj}^n\},$$

where

$$h(\lambda) = \frac{F(d\lambda)}{G(d\lambda)},$$

and $\{s_{kj}^n\}$ and D_n **are matrix**/dies determined by formulas (4.7) and (4.10).

Let us assume that the density h (λ) = F (dλ) /G (dλ) is permissible. On the strength of the condition of weak convergence (4.9) we **have**

$$\lim_{n\to\infty} \int h(\lambda) H_{kj}^{(n)}(d\lambda) = \int h(\lambda) H_{kj}(d\lambda), \quad k, j = 1, \ldots, N,$$

and, furthermore,

$$D_n \{s_{kj}^n\} D_n = \left\{ \int \frac{\psi_k^n(\lambda)\overline{\psi_j^n(\lambda)}}{\|\psi_k^n\| \cdot \|\psi_j^n\|} G(d\lambda) \right\}^{-1} = \left\{ \int H_{kj}^n(d\lambda) \right\}^{-1}.$$

Consequently, under the further condition of the nondegeneracy of the matrix/die

$$H=\left\{\int H_{kl}(d\lambda)\right\},$$

that subsequently let us call the nondegeneracy of measure H (dλ), we have

$$\lim_{n\to\infty} D_n \{s^n_{kl}\} D_n = H^{-1}.$$

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In summation, we obtain, that

$$\lim D_n \{\sigma^n_{kl}\} D_n = H^{-1}\left[\int h(\lambda)\, H(d\lambda)\right] H^{-1}. \qquad (4.11)$$

Thus,

under the condition, when the matrix measures H^n (dλ) with the components of form (4.8) weakly converge to certain measure H (dλ), whereupon matrix/die $H=\int H(d\lambda)$ is nondegenerate, and density h (λ) = F (dλ) /G (dλ) is permissible, for a correlation matrix/die $\{\sigma^n_{kl}\}$ best estimators occurs the following relationship/ratio:

$$\{\sigma^n_{kl}\} \sim D_n^{-1}\left[H^{-1}\int h(\lambda)\, H(d\lambda)\, H^{-1}\right] D_n^{-1}$$

(where the diagonal matrix/die D_n is determined by formula (4.10)), and, in particular,

$$\sigma^n_{kk} = M_\theta [\hat{\alpha}^n_k - \alpha_k]^2 = O\left\{\|\psi^n_k\|_G^{-2}\right\}, \quad k=1, \ldots, N.$$

2. Asymptotics of the errors in best estimators (discrete time). Will be found below explicit asymptotic formulas of type (4.11) for a correlation matrix/die $\{\sigma^n_{kl}\}$ the best estimators $\tilde{\alpha}^n_1, \ldots, \tilde{\alpha}^n_N$, constructed according to the observations random process ξ (t) on interval of $0 \leq t \leq n$ (time t varies discretely).

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In this case, as a rule, will be assumed that there is the continuous spectral density f (λ), that satisfies the condition

$$c_1 \leqslant f(\lambda) \leqslant c_2 \qquad (4.12)$$

(for some positive C_1 and C_2), and functions θ_1 (t), ..., θ_N (t) are such, that

$$\begin{gathered}\lim_{n\to\infty} \sum_0^n \theta_k(t)^2 = \infty \\ \max_{0\leqslant t\leqslant n} |\theta_k(t)| = o\left\{\sqrt{\sum_0^n \theta_k(t)^2}\right\}\end{gathered} \qquad (4.13)$$

(k - 1, ..., N) and for all k, j = 1, ..., N with each s there is the limit

$$\lim_{n\to\infty} \frac{\sum_0^n \theta_k(t+s)\,\theta_j(t)}{\sqrt{\sum_0^n \theta_k(t)^2}\ \sqrt{\sum_0^n \theta_j(t)^2}} = R_{kj}(s). \qquad (4.14)$$

Easy to see that the matrix function

$$R(t) = \{R_{kj}(t)\}, \quad -\infty < t < \infty,$$

is positively determined in the sense that for any moments of time t_1, t_2, ... and any real C_1 C_2, ...

$$\sum_{p,\,q} c_p c_q R_{k_p k_q}(t_p - t_q) \geqslant 0.$$

As is known [1], this matrix function allow/assumes

$$R(t) = \int e^{i\lambda t} H^0(d\lambda), \qquad (4.15)$$

where $H^0(d\lambda) = \{H^0_{kj}(d\lambda)\}$ - the positively determined matrix measure, i.e., components $H^0_{kj}(d\lambda)$ they are complex-valued measures (limited variation) and matrix/die $H^0(\Delta) = \{H^0_{kj}(\Delta)\}$ positively determined with any measurable multitude Δ.

FOOTNOTE [1]. See, for example, [22], page 30. ENDFOOTNOTE.

Let us name H^0 $(d\lambda)$ spectral measure [2] vector function $\{\theta_1(t), \theta_2(t), \ldots, \theta_N(t)\}$. Let us assume that H^0 $(d\lambda)$ is not degenerated (i.e. is not degenerated matrix/die $H_0 = \int H^0(d\lambda)$).

FOOTNOTE [2]. This concept was successfully used by Grenander and Rozenblat during research of the asymptotic efficiency of the estimations of least squares - see, for example, [28]; comp. also with our theorem 10. ENDFOOTNOTE.

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On the strength of condition (4.13) with $n \longrightarrow \infty$

$$\frac{\sum_{0}^{n} \theta_k(t+s)\,\theta_j(t)}{\sqrt{\sum_{0}^{n} \theta_k(t)^2}\,\sqrt{\sum_{0}^{n} \theta_j(t)^2}} \sim \frac{\sum_{0}^{n-s} \theta_k(t+s)\,\theta_j(t)}{\sqrt{\sum_{0}^{n} \theta_k(t)^2}\,\sqrt{\sum_{0}^{n} \theta_j(t)}} =$$

$$= \int e^{i\lambda s} \frac{\theta_k^n(\lambda)\,\overline{\theta_j^n(\lambda)}}{\|\theta_k^n\|_{G^0}\,\|\theta_j^n\|_{G^0}}\, G^0(d\lambda) = \int e^{i\lambda s} H_{kj}^{0,n}(d\lambda),$$

where

$$\theta_k^n(\lambda) = \sum_0^n e^{i\lambda t}\theta_k(t), \quad k = 1, \ldots, N,$$

$$G^0(d\lambda) = \frac{1}{2\pi} d\lambda$$

and

$$H_{kj}^{0,n}(d\lambda) = \frac{\theta_k^n(\lambda)\,\overline{\theta_j^n(\lambda)}}{\|\theta_k^n\|_{G^0}\|\theta_j^n\|_{G^0}} G^0(d\lambda), \quad k, j = 1, \ldots, N.$$

Consequently, with each t

$$R_{kj}(t) = \int e^{i\lambda t} H_{kj}^0(d\lambda) = \lim_{n\to\infty} \int e^{i\lambda t} H_{kj}^{0,n}(d\lambda),$$

that the equivalently weak convergence of the sequence of measures $H_{kj}^{0,n}(d\lambda)$ with n ⟶ ∞ to measure $H_{kj}^0(d\lambda)$:

$$H_{kj}^{0,n} \Rightarrow H_{kj}^0 (k, j = 1, \ldots, N).$$

Thus, spectral measure H^0 (dλ) functions { θ_1 (t), ..., θ_N (t) }, determined from formula (4.15), coincides with matrix measure in formula (4.11), which corresponds to the pseudospectral measure G^0 (dλ) = 1/2π dλ.

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In summation, we obtain an asymptotic formula of type (4.11) for a correlation matrix/die $\{\sigma_{kl}^n\}$ the estimations of least squares - the best estimators, which correspond to pseudospectral measure $G^0(d\lambda) = \frac{1}{2\pi} d\lambda$, namely,

at the H^0-permissible [1] spectral density

$$\lim_{n\to\infty} D_n^0 \{\sigma_{kl}^n\} D_n^0 = 2\pi R(0)^{-1} \int f(\lambda)\, H^0(d\lambda)\, R(0)^{-1}, \quad (4.16)$$

where $R(0) = \int H^0(d\lambda)$, and D_n^0 there is a diagonal matrix/die of the form

$$D_n^0 = \begin{vmatrix} \sqrt{\sum_0^n \theta_1(t)^2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\sum_0^n \theta_N(t)^2} \end{vmatrix}. \quad (4.17)$$

FOOTNOTE [1]. See determination on page 353. ENDFOOTNOTE.

Let us turn now to best estimators $\tilde{\alpha}_1^n, \ldots, \tilde{\alpha}_N^n$, corresponding to

the pseudospectral measure G (dλ) with the density

$$g(\lambda)=\frac{1}{2\pi|Q(e^{i\lambda})|^2}, \quad (4.18)$$

where $Q(z)=\sum_{k=0}^{m} q_k z^k$ - the polynomial with real coefficients, which does not have as zeros in unit circle $|z|\leqslant 1$.

Let us find solution of $\psi(\lambda)\in L_{[0,n]}(G)$ the equation

$$\theta(t)=\int e^{-i\lambda t}\psi(\lambda)\frac{d\lambda}{2\pi|Q(e^{i\lambda})|^2}, \quad 0\leqslant t\leqslant n. \quad (4.19)$$

Let us consider the function θ (t), determined with all t by equality (4.19). With 0 ≤ t ≤ n it coincides with original function θ (t), 0 ≤ t ≤ n.

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If we introduce translation operator Δ: Δθ (t) = θ (t + 1) and the operator

$$Q(\Delta)=\sum_{k=0}^{m} q_k\Delta^k,$$

that let us have the following relationship/ratios:

$$Q(\Delta)\theta(t)=\int [Q(\Delta)e^{-i\lambda t}]\psi(\lambda)\frac{d\lambda}{2\pi|Q(e^{i\lambda})|^2}=$$
$$=\frac{1}{2\pi}\int e^{-i\lambda t}\frac{\psi(\lambda)}{Q(e^{i\lambda})}d\lambda=0$$

with t < 0, since Q (z) - the external **function even** $\psi(\lambda)/Q(e^{i\lambda})$ **enters in subspace** $L_{(0,\infty)}(G^0)$.

In the case, when $n\geqslant m$, **obtained equation**

$$Q(\Delta)\theta(t)=\sum_{k=0}^{m} q_k\theta(t+k)=0,\quad t=-1,\ -2,\ \ldots,$$

makes it possible to consecutively determine the values of the **function** $\theta(t)$ **with t = - 1, -2, ...,** namely:

$$\theta(-1)=-\frac{1}{q_0}\sum_{k=1}^{m} q_k\theta(k-1),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\theta(-j)=-\frac{1}{q_0}\sum_{k=1}^{m} q_k\theta(k-j),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

In perfect analogy it is possible to determine values θ (t) with t = n + 1, n + 2, ... from the equation

$$Q(\Delta^{-1})\theta(t) = \sum_{k=0}^{m} q_k \theta(t-k) = 0; \quad t = n+1,\ n+2,\ \ldots,$$

namely:

$$\theta(n+1) = -\frac{1}{q_0}\sum_{k=1}^{m} q_k \theta(n+1-k),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\theta(n+j) = -\frac{1}{q_0}\sum_{k=1}^{m} q_k \theta(n+j-k),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

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Further,

$$Q(\Delta^{-1})Q(\Delta)\theta(t)=\int[Q(\Delta^{-1})Q(\Delta)e^{-i\lambda t}]\psi(\lambda)\frac{d\lambda}{2\pi|Q(e^{i\lambda})|^2}=$$
$$=\frac{1}{2\pi}\int e^{-i\lambda t}\psi(\lambda)\,d\lambda,\quad -\infty<t<\infty,$$

from which it is clear that the solution $\psi(\lambda)$ equation (4.19) is

$$\psi(\lambda)=\sum_0^n e^{i\lambda t}z(t), \tag{4.20}$$

where

$$z(t)=Q(\Delta^{-1})Q(\Delta)\theta(t),\quad -\infty<t<\infty.$$

(Let us note that with $m \leqslant t \leqslant n - m$ the function $z(t) = Q(\Delta^{-1})Q(\Delta)\theta(t)$ is determined from the initially assigned function $\theta(t)$).

Further, it is easy to see that

$$\sum_{t=0}^{n} e^{i\lambda t} Q(\Delta^{-1}) Q(\Delta)\,\theta(t) = \sum_{t=0}^{n} e^{i\lambda t} \sum_{k,\,j=0}^{m} q_k q_j\,\theta(t-k+j) =$$
$$= \sum_{k,\,j=0}^{m} q_k q_j e^{i\lambda(k-j)} \sum_{t=0}^{n} e^{i\lambda(t-k+j)}\theta(t-k+j) =$$
$$= |Q(e^{i\lambda})|^2 \sum_{0}^{n} e^{i\lambda s}\theta(s) + r(\lambda),$$

where in the trigonometric sum

$$r(\lambda) = \sum_{-m}^{m} a_k e^{i\lambda k} + \sum_{n-m}^{n+m} b_k e^{i\lambda k}$$

the coefficients

$$a_k = \sum_{j=0}^{m-1} c'_{kj}\theta(j),$$
$$b_k = \sum_{j=0}^{m-1} c''_{kj}\theta(n-j)$$

are the linear combinations of values $\theta(j)$, $j = 0, \ldots, m-1$ and $\theta(k)$, $k = n-1, \ldots, n-m+1$.

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It is evident that

the solution $\psi(\lambda)$ equation (4.19) has the following structure:

$$\psi(\lambda)=|Q(e^{i\lambda})|^2\tilde{\theta}(\lambda)+r(\lambda), \qquad (4.21)$$

where

$$\tilde{\theta}(\lambda)=\sum_0^n e^{i\lambda t}\theta(t).$$

Let us use the obtained results to functions $\theta_1(t), \ldots, \theta_N(t)$. As it follows from common/general/total formula (4.21),

$$\psi_k^n(\lambda)=|Q(e^{i\lambda})|^2\tilde{\theta}_k^n(\lambda)+r_k^n(\lambda),$$

where under condition (4.13)

$$\|r_k^n\|_G=o\{\|\theta_k^n\|_{G^0}\}, \quad k=1, \ldots, N$$

(recall that $G^0(d\lambda)=d\lambda/2\pi$ and $G(d\lambda)=1/2\pi|Q(e^{i\lambda})|^2$). Therefore under the condition for existence (matrix) of the spectral measure $H^0(d\lambda)$ of functions $\{\theta_1(t), \ldots, \theta_N(t)\}$ for any limited and continuous function $\phi(\lambda)$ we have

$$\lim_{n\to\infty}\int\varphi(\lambda)\frac{\psi_k^n(\lambda)\overline{\psi_l^n(\lambda)}}{\|\theta_k^n\|_{G^0}\|\theta_l^n\|_{G^0}}G(d\lambda)=$$
$$=\lim_{n\to\infty}\int\varphi(\lambda)|Q(e^{i\lambda})|^2\frac{\tilde{\theta}_k^n(\lambda)}{\|\theta_k^n\|_{G^0}}\frac{\overline{\tilde{\theta}_l^n(\lambda)}}{\|\theta_l^n\|_{G^0}}G^0(d\lambda)=$$
$$=\int\varphi(\lambda)|Q(e^{i\lambda})|^2H_{kl}^0(d\lambda) \qquad (4.22)$$

and, in particular,

$$\lim_{n\to\infty}\frac{\langle\psi_k^n, \psi_l^n\rangle_G}{\|\theta_k^n\|_{G^0}\|\theta_l^n\|_{G^0}}=\int|Q(e^{i\lambda})|^2H_{kl}^0(d\lambda).$$

It is evident that the matrix measure $H(d\lambda)=\{H_{kl}(d\lambda)\}$ in asymptotic

formula (4.11) has the components

$$H_{kl}(d\lambda) = \left[\int |Q(e^{i\lambda})|^2 H^0_{kk}(d\lambda)\right]^{-1/2} |Q(e^{i\lambda})|^2 H^0_{kl}(d\lambda) \times \\ \times \left[\int |Q(e^{i\lambda})|^2 H^0_{ll}(d\lambda)\right]^{-1/2}, \quad (4.23)$$

a diagonal matrix/die D_n is such, that (see (4.10), (4.17))

$$\lim_{n\to\infty} D^0_n D^{-1}_n = \\ = \left|\begin{matrix} \sqrt{\int |Q(e^{i\lambda})|^2 H^0_{11}(d\lambda)} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\int |Q(e^{i\lambda})|^2 H^0_{NN}(d\lambda)} \end{matrix}\right|^{-1}.$$

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In summation, we obtain the following result.

Theorem 8. Let the functions $\theta_1(t), \ldots, \theta_N(t)$ have the nondegenerate spectral measure $H^0(d\lambda)$ and satisfy condition (4.13), but the spectral density $f(\lambda)$ is h^0- permissible. Then at any spectral density $g(\lambda)$ of the form

$$g(\lambda) = \frac{1}{2\pi |Q(e^{i\lambda})|^2} \quad (4.24)$$

for a correlation matrix/die $\{\sigma^n_{kl}\}$ pseudo-best estimations $\tilde{\alpha}^n_1, \ldots, \tilde{\alpha}^n_N$ occurs the following asymptotic relationship/ratio [1]:

$$\lim_{n\to\infty} D^0_n \{\sigma^n_{kl}\} D^0_n = 2\pi\left[\int \frac{1}{g(\lambda)} H^0(d\lambda)\right]^{-1}\left[\int h(\lambda)\frac{1}{g(\lambda)} H^0(d\lambda)\right]\left[\int \frac{1}{g(\lambda)} H^0(d\lambda)\right]^{-1}, \tag{4.25}$$

where

$$h(\lambda) = \frac{f(\lambda)}{g(\lambda)},$$

a D^0_n is diagonal matrix/die (4.17).

FOOTNOTE [1]. It is necessary to say that relationship/ratio (4.25) occurs for any continuous and positive spectral density g (λ) (see Yu. A. Rozanov, M. O. Kozlov, the asymptotically efficient evaluation of the coefficients of the regression, PAS of USSR 1 (1969)), but from the viewpoint of application/appendices, of course, it suffices to be bounded to spectral densities of type (4.18). ENDFOOTNOTE.

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Let us turn to perform linear unbiased estimates, using for them the same designations

$$\tilde{\alpha}_k^n = \int \varphi_k^n(\lambda)\,\Phi(d\lambda), \qquad k = 1, \ldots, N,$$

as for the examined above best estimators. Let us show that under condition (4.12) for the correlation matrix/die

$$\{\sigma_{kl}^n\} = \left\{ \int \varphi_k^n(\lambda)\,\overline{\varphi_l^n(\lambda)}\,f(\lambda)\,d\lambda \right\}$$

occurs the following asymptotic lower limit:

$$\varliminf_{n\to\infty} D_n^0 \{\sigma_{kl}^n\} D_n^0 \geqslant 2\pi \left\{ \int \frac{1}{f(\lambda)} H^0(d\lambda) \right\}^{-1}. \qquad (4.26)$$

For a proof we will use one common/general/total inequality. Specifically, if $x_1, \ldots, x_m$ - arbitrary cell/elements hilbert space, a $y_1, \ldots, y_n$- linearly independent cell/elements, then

$$\{\langle x_i, x_l\rangle\} \geqslant \{\langle \tilde{x}_i, \tilde{x}_l\rangle\} = \{\langle \tilde{x}_i, y_k\rangle\}\{\langle y_k, y_l\rangle\}^{-1}\{\langle \tilde{x}_l, y_l\rangle\}^* =$$
$$= \{\langle x_i, y_k\rangle\}\{\langle y_k, y_l\rangle\}^{-1}\{\langle x_l, y_l\rangle\}^*, \qquad (4.27)$$

where $\tilde{x}_1, \ldots, \tilde{x}_n$ they indicate the projections of parent elements $x_1, \ldots, x_n$ on the subspace, generated by cell/elements $y_1, \ldots, y_n$, and they are located through the formulas

$$\tilde{x}_i = \sum_{k,l} c_{kl}\langle x_i, x_l\rangle y_k, \qquad i = 1, \ldots, m,$$
$$\{c_{kl}\} = \{\langle y_k, y_l\rangle\}^{-1}.$$

Applying inequality (4.27) to cell/elements $\varphi_1^n(\lambda), \ldots, \varphi_N^n(\lambda)$ and $\psi_1^n(\lambda)/h(\lambda), \ldots, \psi_N^n(\lambda)/h(\lambda)$ from hilbert space $L_{(-\infty,\infty)}(F)$, we obtain, that

$$\{\sigma_{kl}^n\} = \{\langle\varphi_i^n, \varphi_l^n\rangle_F\} \geqslant$$
$$\geqslant \{\langle\varphi_i^n, \psi_k^n\rangle_G\} \left\{ \int \frac{2\pi g(\lambda)}{h(\lambda)} \psi_k^n(\lambda) \overline{\psi_l^n(\lambda)} G^0(d\lambda) \right\}^{-1} \{\langle\varphi_l^n, \psi_l^n\rangle_G\}^*,$$

since the cell/elements $\psi_1^n/h, \ldots, \psi_N^n/h$ are linearly independent, and under condition (4.12) the cell/elements $\varphi_1^n, \ldots, \varphi_N^n$ enter in hilbert space $L_{(-\infty,\infty)}(G)$ and

$$\left\langle\varphi_i^n(\lambda), \frac{\psi_k^n(\lambda)}{h(\lambda)}\right\rangle_F = \langle\varphi_i^n, \psi_k^n\rangle_G, \quad i, k = 1, \ldots, N,$$
$$\left\langle\frac{\psi_k^n(\lambda)}{h(\lambda)}, \frac{\psi_j^n(\lambda)}{h(\lambda)}\right\rangle_F = \int \frac{g(\lambda)}{f(\lambda)} \psi_k^n(\lambda) \overline{\psi_j^{(n)}(\lambda)} G(d\lambda),$$
$$k, j = 1, \ldots, N.$$

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(Recall that h (λ) = f (λ) /g (λ) and $g(\lambda) = 1/2\pi |Q(e^{i\lambda})|^2$.)

Furthermore, from spectral representation (4.2) and condition of the nondisplacement of estimations $\tilde{\alpha}_1^n, \ldots, \tilde{\alpha}_N^n$ we have

$$\alpha_i = M_0 \tilde{\alpha}_i^n = \int \varphi_i^n(\lambda)\, M_0 \Phi(d\lambda) =$$
$$= \int \varphi_i^n(\lambda) \sum_{k=1}^{N} \alpha_k \overline{\psi_k^n(\lambda)}\, G(d\lambda), \qquad i = 1, \ldots, N,$$

whence in view of the arbitrariness of coefficients $\alpha_1, \ldots, \alpha_n$ it follows that

$$\{\langle \varphi_i^n, \psi_k^n \rangle_G\} = E$$

(E - unit matrix). Consequently,

$$\{\sigma_{kj}^n\} \geqslant \left\{ \int \frac{g(\lambda)}{f(\lambda)} \psi_k^n(\lambda) \overline{\psi_j^n(\lambda)}\, G(d\lambda) \right\}^{-1}.$$

Since under condition (4.12), when $1/f(\lambda) \geqslant c > 0$, occurs inequality $\int \frac{1}{f(\lambda)} H^0(d\lambda) \geqslant c \int H^0(d\lambda)$, then together with matrix/die $H^0 = \int H^0(d\lambda)$ will be nondegenerate matrix/die $\int \frac{1}{f(\lambda)} H^0(d\lambda)$, and from relationship/ratio (4.22) we obtain, that

$$\lim_{n\to\infty} D_n^0 \{\sigma_{kj}^n\} D_n^0 \geqslant$$
$$\geqslant \lim_{n\to\infty} \left\{ \int \frac{g(\lambda)}{f(\lambda)} \frac{\psi_k^n(\lambda)}{\|\theta_k^n\|_{G^0}} \frac{\overline{\psi_j^n(\lambda)}}{\|\theta_j^n\|_{G^0}} G(d\lambda) \right\}^{-1} =$$
$$= 2\pi \left\{ \int \frac{1}{f(\lambda)} H^0(d\lambda) \right\}^{-1}.$$

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Thus, under condition (4.12) for any unbiased linear estimates $\tilde{\alpha}_1^n, \ldots, \tilde{\alpha}_N^n$ with correlation matrix/die $\{\sigma_{kl}^n\}$ occurs inequality (4.26).

Let us show that for best estimators (with correlation matrix/die $\{s_{kl}^n\}$) this inequality is converted into precise equality.

It is obvious, sufficient to show that

$$\lim_{n\to\infty} D_n^0 \{s_{kl}^n\} D_n^0 \leqslant 2\pi \left[\int \frac{1}{f(\lambda)} H^0(d\lambda)\right]^{-1}. \qquad (4.28)$$

Let us demonstrate the validity of inequality (4.28). Under condition (4.12) continuous positive function 1/f (λ) can be how conveniently accurately approximated by the trigonometric (positive) polynomials, always representable in the form $2\pi|Q(e^{i\lambda})|^2$. Therefore in formula (4.25) it is possible to select the spectral density

$$g(\lambda) = \frac{1}{2\pi |Q(e^{i\lambda})|^2},$$

how conveniently differing little from spectral density f (λ). Since always $\{s_{kl}^n\} \leqslant \{\sigma_{kl}^n\}$,

$$\lim_{n\to\infty} D_n^0 \{s_{kl}^n\} D_n^0 \leqslant$$
$$\leqslant 2\pi \left[\int \frac{1}{g(\lambda)} H^0(d\lambda)\right]^{-1} \left[\int \frac{f(\lambda)}{g(\lambda)^2} H^0(d\lambda)\right] \left[\int \frac{1}{g(\lambda)} H^0(d\lambda)\right]^{-1},$$

where g (λ) can be selected how convenient to close to f (λ), and therefore must be correctly and inequality (4.28).

From relationship/ratios (4.26) and (4.28) escape/ensues the following result.

Theorem 9. Under condition (4.12) in the case of the existence of the nondegenerate spectral measure H^0 (dλ) for a correlation matrix/die $\{s_{kj}^n\}$ the best unbiased estimates occurs the following asymptotic formula:

$$\lim_{n\to\infty} D_n^0 \{s_{kj}^n\} D_n^0 = 2\pi \left[\int \frac{1}{f(\lambda)} H^0(d\lambda) \right]^{-1}. \qquad (4.29)$$

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The obtained above results make it possible to deduce the conditions of the asymptotic efficiency of best estimators.

Recall that for the unbiased estimate $\tilde{\alpha}$ the parameter α, which allow/assumes the best unbiased estimate $\hat{\alpha}$ with the minimum dispersion, the efficiency is defined as relation $\frac{M(\hat{\alpha}-\alpha)^2}{M(\tilde{\alpha}-\alpha)^2}$. Accordingly estimation $\tilde{\alpha}_1^n, \ldots, \tilde{\alpha}_N^n$ they are called asymptotically efficient, if

$$\lim_{n\to\infty} \frac{M(\hat{\alpha}_k^n - \alpha_k)^2}{M(\tilde{\alpha}_k^n - \alpha_k)^2} = 1, \qquad k = 1, \ldots, N.$$

Let us show that in that which is examine/considered by us the

case the asymptotic efficiency of best estimators $\tilde{\alpha}_1^n, \ldots, \tilde{\alpha}_N^n$ is equivalent to the condition

$$\lim_{n\to\infty} D_n^0 \{s_{kj}^n\} D_n^0 = \lim D_n^0 \{\sigma_{kj}\} D_n^0. \qquad (4.30)$$

In fact, in examine/considered by us the case there are the nondegenerate maximum matrix/dies

$$\{a_{kj}\} = \lim D_n^0 \{s_{kj}^n\} D_n^0, \quad \{b_{kj}\} = \lim D_n^0 \{\sigma_{kj}^n\} D_n^0,$$

on the strength of positive certainty of which all the diagonal cell/elements

$$a_{kk} = \lim d_k^n s_{kk}^n d_k^n \quad \textbf{and} \quad b_{kk} = \lim d_k^n \sigma_{kk}^n d_k^n$$

are different from 0 (here d_k^n the diagonal cell/elements of the normalization matrix/die D_n^0). Condition (4.30) means that $\{a_{kj}\} = \{b_{kj}\}$ and, in particular, that

$$\lim (d_k^n)^2 s_{kk}^n = \lim (d_k^n)^2 \sigma_{kk}^n \neq 0.$$

Consequently, under condition (4.30) is satisfied the condition of the asymptotic efficiency: $\lim s_{kk}^n/\sigma_{kk}^n = 1, k = 1, \ldots, N$. In turn, this condition means that $\sigma_{kk}^n - s_{kk}^n = o(\sigma_{kk}^n)$, where $\sigma_{kk}^n \sim b_{kk}(d_k^n)^{-2}$, and since the matrix/die $\{\sigma_{kj}^n - s_{kj}^n\}$ is nonnegative, with any k, j = 1, ..., N

$$|\sigma_{kj}^n - s_{kj}^n| \leqslant (\sigma_{kk}^n - s_{kk}^n)^{1/2} (\sigma_{jj}^n - s_{jj}^n)^{1/2} = o(\sqrt{\sigma_{kk}^n \sigma_{jj}^n}).$$

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Consequently, with the asymptotic efficiency of the estimations

$$b_{kj} = \lim d_k^n [s_{kj}^n + (\sigma_{kj}^n - s_{kj}^n)] d_j^n = \\ = a_{kj} + \lim d_k^n (\sigma_{kj}^n - s_{kj}^n) d_j^n = a_{kj}, \quad k, j = 1, \ldots, N.$$

in question i.e. occurs relationship/ratio (4.30).

We will use asymptotic formulas (4.25) and (4.29). Let us consider the standardized/normalized matrix measure of form (4.23)

$$H^*(d\lambda) = \left[\int \frac{1}{g(\lambda)} H^0(d\lambda)\right]^{-1/2} \frac{1}{g(\lambda)} H^0(d\lambda) \left[\frac{1}{g(\lambda)} H^0(d\lambda)\right]^{-1/2},$$

where g (λ) - spectral density. Then under condition, when are valid formulas (4.25) and (4.29), for an asymptotic efficiency necessary and it is sufficient in order that would be implemented (equivalent (4.30)) the equality

$$\int h(\lambda) H^*(d\lambda) = \left[\int \frac{1}{h(\lambda)} H^*(d\lambda)\right]^{-1}, \qquad (4.31)$$

where h (λ) = f (λ) /g (λ).

Let us show that it is implemented when and only when scalar matrix/die $\frac{f(\lambda)}{g(\lambda)} E$ is constant almost everywhere relatively $H^*(d\lambda)$: $\frac{f(\lambda)}{g(\lambda)} E = C$ $(C = \{c_{kj}\})$. more precise,

$$\left(\frac{f(\lambda)}{g(\lambda)} E - C\right) H^*(d\lambda) = 0, \qquad (4.32)$$

where E is single, $C = \{c_{kj}\}$ - certain constant of matrix/die.

For a proof let us introduce the densities

$$h_{kj}(\lambda)=\frac{H_{kj}(d\lambda)}{m(d\lambda)}, \quad k, j=1, \ldots, N$$

(relative to certain measure m (dλ)) and present positive matrix/die $\{h_{kj}(\lambda)\}$ in the form of the product

$$\{h_{kj}(\lambda)\}=\{\varphi_{ik}(\lambda)\}\{\varphi_{kj}(\lambda)\},$$

where $\{\varphi_{ik}\}$ – positive square root from matrix/die $\{h_{ij}\}$.

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Vector functions $\varphi_i(\lambda)=\{\varphi_{i1}(\lambda), \ldots, \varphi_{iN}(\lambda)\}$ can be considered as cell/elements the hilbert space, in which the scalar product of cell/elements $\varphi=\{\varphi_1, \ldots, \varphi_N\}$ and $\psi=\{\psi_1, \ldots, \psi_N\}$ is determined by the formula

$$\langle\varphi, \psi\rangle=\int \sum_{k=1}^{N} \varphi_k(\lambda) \overline{\psi_k(\lambda)}\, m(d\lambda).$$

If we consider tha cell/elements

$$\sqrt{h}\, \varphi_i=\{\sqrt{h(\lambda)}\, \varphi_{i1}, \ldots, \sqrt{h(\lambda)}\, \varphi_{iN}\}, \quad i=1, \ldots, N,$$

and

$$\frac{1}{\sqrt{h}} \varphi_i=\left\{\frac{1}{\sqrt{h(\lambda)}} \varphi_{i1}, \ldots, \frac{1}{\sqrt{h(\lambda)}} \varphi_{iN}(\lambda)\right\}, \quad i=1, \ldots, N,$$

that will seen that

$$\{\langle\sqrt{h}\, \varphi_i, \sqrt{h}\, \varphi_j\rangle\}=\int h(\lambda) H^{*}(d\lambda)$$

and

$$\left\{\left\langle \frac{1}{\sqrt{h}}\varphi_i, \frac{1}{\sqrt{h}}\varphi_l\right\rangle\right\} = \int \frac{1}{h(\lambda)} H^*(d\lambda),$$

a furthermore,

$$\left\{\left\langle \sqrt{h}\varphi_i, \frac{1}{\sqrt{h}}\varphi_l\right\rangle\right\} = \int H^*(d\lambda) = E.$$

Consequently, equality (4.31) can be rewritten in the following form:

$$\begin{aligned}\{\langle\sqrt{h}\varphi_i, \sqrt{h}\varphi_l\rangle\} = \\ = \left\{\left\langle \sqrt{h}\varphi_i, \frac{1}{\sqrt{h}}\varphi_k\right\rangle\right\}\left\{\left\langle \frac{1}{\sqrt{h}}\varphi_k, \frac{1}{\sqrt{h}}\varphi_l\right\rangle\right\}^{-1} \times \\ \times \left\{\left\langle \sqrt{h}\varphi_j, \frac{1}{\sqrt{h}}\varphi_l\right\rangle\right\}.\end{aligned}$$

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Generally speaking, instead of the equal sign here must be inequality sign (see (4.27)), and equality occurs if and only if the vectors $\sqrt{h(\lambda)}\varphi_1(\lambda), \ldots\ldots, \sqrt{h(\lambda)}\varphi_N(\lambda)$ are the linear combinations of vectors $\frac{1}{\sqrt{h(\lambda)}}\varphi_1(\lambda), \ldots, \frac{1}{\sqrt{h(\lambda)}}\varphi_N(\lambda)$, i.e., when for certain constant matrix/die $C = \{c_{ij}\}$ almost everywhere relative to m (dλ)

$$\sqrt{h(\lambda)}\varphi_{ik}(\lambda) = \sum_{j=1}^{N} c_{ij}\frac{1}{\sqrt{h(\lambda)}}\varphi_{jk}(\lambda), \quad k = 1, \ldots, N$$
$$(i = 1, \ldots, N).$$

Hence we obtain, that

$$h(\lambda)\sum_{k=1}^{N}\varphi_{lk}(\lambda)\varphi_{kl}(\lambda)=h(\lambda)h_{il}(\lambda)=$$
$$=\sum_{k=1}^{N}\left[\sum_{l=1}^{N}c_{il}\varphi_{lk}(\lambda)\right]\varphi_{kl}(\lambda)=\sum_{k}c_{ik}h_{kl}(\lambda),$$

and, thus, almost everywhere relative to m (dλ)

or

$$\left(\frac{f(\lambda)}{g(\lambda)}E-\{c_{ik}\}\right)\{h_{kl}(\lambda)\}=0,$$
$$\left(\frac{f(\lambda)}{g(\lambda)}E-C\right)H^{\circ}(d\lambda)=0,$$

Q. E. D. From equality (4.32) it is easy to deduce the following result.

Theorem 10. Under condition (4.12) for the asymptotic efficiency of the best estimators of coefficients $\alpha_1, \ldots, \alpha_N$ in expansion (4.1) (corresponding to spectral density g (λ)) it is necessary and sufficiently in order that matrix function $\frac{f(\lambda)}{g(\lambda)}E$ would be constant almost everywhere relative to H^0 (dλ).

3. Asymptotic behavior of the errors in best estimators (continuous time). Let us consider the case continuous t, when the

discussion concerns best estimators $\tilde{\alpha}_1^n, \ldots, \tilde{\alpha}_N^n$ according to the observations of process ξ (t) in intervals of 0 ≤ t ≤ τ, where $\tau = \tau_n \to \infty$. In which measure asymptotic formulas for the correlation matrix/dies $\{\sigma_{kl}^n\}$ best estimators, obtained by us in the case of the discrete time t, do extend to the continuous case?

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Of course, the spectral measure $H^0(d\lambda) = \{H_{kl}^0(d\lambda)\}$ vector function $\{\theta_1(t), \ldots, \theta_N(t)\}$ is defined in perfect analogy with that, as this is done in the discrete case, namely the components $H_{kl}^0(d\lambda)$ are determined from the relationship/ratios

$$R_{kj}(s) = \lim_{n\to\infty} \frac{\int_0^\tau \theta_k(t+s)\,\theta_j(t)\,dt}{\sqrt{\int_0^\tau \theta_k(t)^2\,dt}\sqrt{\int_0^\tau \theta_j(t)^2\,dt}} = \int e^{i\lambda s} H_{kj}^0(d\lambda), \quad k, j = 1, \ldots, N \qquad (4.33)$$

(in this case additionally it is assumed that the function $\{R_{kj}(s)\}$ is continuous in 0).

Let

$$\lim_{n\to\infty} \int_0^\tau \theta_k(t)^2\,dt = \infty$$

and

$$\max_{0 \leq t \leq \tau} |\theta_k(t)| = o\left\{\sqrt{\int_0^\tau \theta_k(t)^2\, dt}\right\},$$
$$k = 1, \ldots, N,$$

a spectral density $f(\lambda)$ H^0- be permissible [1], whereupon H⁰ (dλ) is not degenerated.

FOOTNOTE [1]. See determination on page 354. ENDFOOTNOTE.

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Then, just as in the case of discrete time, for a correlation matrix/die $\{\sigma^n_{kl}\}$ the estimations of least squares occurs formula (4.16):

$$\lim_{n\to\infty} D^0_n \{\sigma^n_{kl}\} D^0_n = 2\pi R(0)^{-1} \int f(\lambda)\, H^0(d\lambda)\, R(0)^{-1},$$

in which $R(0) = H^0 = \int H^0(d\lambda)$, a D^0_n is a diagonal matrix/die of the form

$$D_n^0 = \begin{Vmatrix} \sqrt{\int_0^\tau \theta_1(t)^2\,dt} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\int_0^\tau \theta_N(t)^2\,dt} \end{Vmatrix}. \quad (4.34)$$

But in the examination of the best estimators, which correspond to the spectral density of the form

$$g(\lambda) = \frac{1}{2\pi\,|Q(i\lambda)|^2}, \quad (4.35)$$

where $Q(z) = \sum_{k=0}^{m} q_k z^k$ – polynomial with real by coefficients, immediately appears the question concerning the legitimacy of passage to the limit in a relationship/ratio of type (4.22), since the function Q (iλ) is not limited when-∞ < λ < ∞. The same question arises also during reference to a formula of type (4.29), since the spectral density f (λ) is integrated and function 1/f (λ) is not limited when-∞ < λ < ∞.

Let us assume that the functions $\theta_1(t), \ldots, \theta_N(t)$ have 2m continuous derivatives, also, for functions $\{y_1(t), \ldots, y_N(t)\}$, where

$$y_k(t) = Q\left(\frac{d}{dt}\right)\theta(t), \quad k = 1, \ldots, N, \quad (4.36)$$

there is a nondegenerate spectral measure, which let us designate $H(d\lambda) = \{H_{kj}(d\lambda)\}$. Furthermore, let us suppose that

$$\max_{0 \leqslant t \leqslant \tau} \left| \frac{d^k}{dt^k} \theta_j(t) \right| = o\left\{ \sqrt{\int_0^\tau y_j(t)^2\, dt} \right\} \qquad (4.37)$$
$$(k = 0, \ldots, 2m;\ j = 1, \ldots, N).$$

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Existence of the spectral measure H (dλ) under condition (4.37) of equivalently waak convergence (to H (dλ)) the sequence of matrix measures $M^n(d\lambda) = \{M^n_{kj}(d\lambda)\}$ with the components

$$M^n_{kj}(d\lambda) = \frac{\tilde{y}^n_k(\lambda)}{\|\tilde{y}^n_k\|_{G^0}} \frac{\overline{\tilde{y}^n_j(\lambda)}}{\|\tilde{y}^n_j\|_{G^0}} G^0(d\lambda), \quad k, j = 1, \ldots, N, \quad (4.38)$$

where

$$\tilde{y}^n_k(\lambda) = \int_0^\tau e^{i\lambda t} y_k(t)\, dt, \qquad k = 1, \ldots, N$$

$(\tau = \tau_n \to \infty)$, and $G^0(d\lambda) = \frac{1}{2\pi} d\lambda$, i.e., for any limited and continuous function φ (λ)

$$\lim_{n \to \infty} \int \varphi(\lambda) M^n(d\lambda) = \int \varphi(\lambda) H(d\lambda).$$

Let us turn to formula (4.3), which describes the best estimators, constructed according to spectral density g (λ) of form (4.35), and let us consider the appropriate integral equation

$$\theta(t) = \int e^{-i\lambda t} \psi(\lambda) \frac{d\lambda}{2\pi |Q(i\lambda)|^2}, \qquad 0 \leqslant t \leqslant \tau \quad (4.39)$$

(type (4.6)), relative to function $\psi(\lambda) \in L_{[0,\tau]}(G)$.

Let the function θ (t), $-\infty < t < \infty$, with all t be defined as the left side in (4.39). Since the polynomial Q (z) in (4.35) is external function (for a lower half-plane), during any function $\psi(\lambda)$ from subspace $L_{[0,\infty)}(G)$ relation $\psi(\lambda)/Q(-i\lambda)$ will be function from $L_{[0,\infty)}(G^0)$ (see §2 of chapter II), and therefore for solution $\psi(\lambda) \in L_{[0,\tau]}(F)$ and the corresponding function θ (t) let us have

$$Q\left(-\frac{d}{dt}\right)\theta(t) = \int e^{-i\lambda t}\frac{\psi(\lambda)}{Q(-i\lambda)}\frac{d\lambda}{2\pi} = 0 \quad \text{with} \quad t<0.$$

For the completely analogous reasons

$$Q\left(\frac{d}{dt}\right)\theta(t) = \int e^{-i\lambda t}\frac{\psi(\lambda)}{Q(i\lambda)}\frac{d\lambda}{2\pi} = 0 \quad \text{with} \quad t>\tau$$

(since $\psi(\lambda)/Q(i\lambda) \in L_{(-\infty,\tau]}(G^0)$).

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It is evident, that solution $\psi(\lambda) \in L_{[0,\tau]}(G)$ integral equation (4.39) is the generalized Fourier transform [1] the generalized function

$$z(t) = Q\left(\frac{d}{dt}\right) Q\left(-\frac{d}{dt}\right) \theta(t) = Q\left(-\frac{d}{dt}\right) Q\left(\frac{d}{dt}\right) \theta(t).$$

FOOTNOTE [1]. See, for example, [6]. ENDFOOTNOTE.

But with $0 < t < \tau$ the left side of equation (4.39) coincides with the initially assigned function θ (t), which has 2m usual derivatives; with $-\infty < t < \tau$ usual function $Q\left(-\frac{d}{dt}\right)\theta(t)$, as it was shown above, it is converted in 0 with $-\infty < t < 0$, and therefore the generalized function $z(t) = Q\left(\frac{d}{dt}\right)\left[Q\left(-\frac{d}{dt}\right)\theta(t)\right]$ with $-\infty < t < \tau$ is obtained by the application/use of a differential operator $Q\left(\frac{d}{dt}\right)$ to the usual sectionally smooth function $Q\left(-\frac{d}{dt}\right)\theta(t)$, equal to 0 with $-\infty < t < 0$ and which has 2m derivatives with $0 < t < \tau$.

Similar pattern for z (t) occurs, also, with $0 < t < \infty$, so that in summation, we obtain the following formula [2] for the generalized function z (t):

$$z(t) = z_0(t) + \sum_{k=0}^{m-1} a'_k \delta^{(k)}(t) + \sum_{k=0}^{m-1} b'_k \delta^{(k)}(t-\tau),$$

where the coefficients a_k' and b_k' essence linear combinations derived not higher than m - 1 of functions $Q\left(-\frac{d}{dt}\right)\theta(t)$and $Q\left(\frac{d}{dt}\right)\theta(t)$ at end-points t = 0 and t = τ, the generalized functions $\delta^{(k)}(t)$ and $\delta^{(k)}(t-\tau)$ essence the k derivatives of δ-functions at the same points t = 0 and t = τ, but z_0 (t) are a usual function of the form

$$z_0(t)=\begin{cases} Q\left(-\frac{d}{dt}\right)Q\left(\frac{d}{dt}\right)\theta(t) & \text{with } 0<t<\tau, \\ 0 \end{cases}$$

with remaining t.

FOOTNOTE [2]. See, for example [22], page 191. ENDFOOTNOTE.

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Now, if we introduce the function

$$y(t)=Q\left(\frac{d}{dt}\right)\theta(t), \quad 0\leqslant t\leqslant\tau,$$

solution $\psi(\lambda)$ equation (4.39) can be presented in the form

$$\psi(\lambda)=\int_0^\tau e^{i\lambda t}\left[Q\left(-\frac{d}{dt}\right)y(t)\right]dt+ \\ +\sum_{k=0}^{m-1}a_k'(i\lambda)^k+\sum_{k=0}^{m-1}b_k'e^{i\lambda\tau}(i\lambda)^k.$$

Integrating in parts, we obtain, that

$$\int_0^\tau e^{i\lambda t}\left[Q\left(-\frac{d}{dt}\right)y(t)\right]dt =$$
$$= Q(i\lambda)\int_0^\tau e^{i\lambda t}y(t)\,dt + \sum_{k=0}^{m-1} a_k''(i\lambda)^k + \sum_{k=0}^{m-1} b_k'' e^{i\lambda\tau}(i\lambda)^k,$$

where the coefficients a_k'' and b_k'' are linear combinations derived not higher than m - 1 of function y (t) at end-points t = 0 and t = τ. By using an obtained equation,

solution $\psi(\lambda)$ equation (4.39) can be presented in the form (comp. (4.21))

$$\psi(\lambda) = Q(i\lambda)\tilde{y}(\lambda) + r(\lambda), \qquad (4.40)$$

where

$$\tilde{y}(\lambda) = \int_0^\tau e^{i\lambda t}\,y(t)\,dt,$$

and the coefficients a_k and b_k in the expression

$$r(\lambda) = \sum_0^{m-1} a_k(i\lambda)^k + \sum_0^{m-1} b_k e^{i\lambda\tau}(i\lambda)^k$$

are the linear combinations of derivatives not above 2m - 1 of original function θ (t), $0 \leqslant t \leqslant \tau$, at end-points t = 0 and t = τ:

$$a_k = \sum_{j=0}^{2m-1} c'_{kj}\theta^{(j)}(0), \quad b_k = \sum_{j=0}^{2m-1} c''_{kj}\theta^{(j)}(\tau).$$

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Will use the obtained results to functions $\psi_1^n(\lambda), \ldots, \psi_N^n(\lambda)$, with the help of which are constructed best estimators $\tilde{a}_1^n, \ldots, \tilde{a}_N^n$ (see (4.6)). According to formula (4.40)

$$\psi_k^n(\lambda) = Q(i\lambda)\,\tilde{y}_k^n(\lambda) + r_k^n(\lambda) \qquad (4.41)$$

(k = 1, ..., N), where under condition (4.37) "residue/remainder" $r_k^n(\lambda)$ satisfies the relationship/ratio

$$|r_k^n(\lambda)| = o\{|\lambda|^{m-1}\|\tilde{y}_k^n\|_{G^*}\},$$

and therefore for pseudospectral measure $G(d\lambda) = \frac{d\lambda}{2\pi|Q(i\lambda)|^2}$ (where Q (z) - the polynomial of degree 2m)

$$\|r_k^n\|_G = o\{\|\tilde{y}_k^n\|_{G^*}\}.$$

From (4.41) it is easy to deduce, that

$$\lim_{n\to\infty}\frac{\|\psi_k^n\|_G}{\|y_k^n\|_{G^0}}=1,$$

and under the condition of the weak convergence of the introduced above (see (4.38) measures $M_{kj}^n(d\lambda)$ we have

$$\lim_{n\to\infty}\int\varphi(\lambda)\frac{\psi_k^n(\lambda)\overline{\psi_j^n(\lambda)}}{\|\psi_k^n\|_G\|\psi_j^n\|_G}G(d\lambda)=$$
$$=\lim_{n\to\infty}\int\varphi(\lambda)\frac{y_k^n(\lambda)}{\|y_k^n\|_{G^0}}\frac{\overline{y_j^n(\lambda)}}{\|y_j^n\|_G}G^0(d\lambda)=$$
$$=\int\varphi(\lambda)H_{kj}(d\lambda),\qquad k,\ j=1,\ \ldots,\ N\quad(4.42)$$

(for any limited and continuous function ϕ (λ). It is evident that the matrix measures $H^n(d\lambda)=\{H_{kj}^n(d\lambda)\}$ with the components

$$H_{kj}^n(d\lambda)=\frac{\psi_k^n(\lambda)\overline{\psi_j^n(\lambda)}}{\|\psi_k^n\|_G\|\psi_j^n\|_G}G(d\lambda),\qquad k,\ j=1,\ \ldots,\ N,$$

weakly are reduced to the spectral measure H (dλ) = $\{H_{kj}(d\lambda)\}$ of vector function $\{y_1(t),\ \ldots,\ y_N(t)\}$.

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Thus,

for a correlation matrix/die $\{\sigma_{kj}^n\}$ best estimators occurs asymptotic formula (4.11), in which H (dλ) there is a spectral measure of functions (4.36), namely:

$$\lim_{n\to\infty}D_n\{\sigma_{kj}^n\}D_n=H^{-1}\left[\int\frac{f(\lambda)}{g(\lambda)}H(d\lambda)\right]H^{-1},\quad(4.43)$$

where $H = \int H(d\lambda)$, D_n is a diagonal matrix/die of the form

$$D_n = \begin{pmatrix} \|\tilde{y}_1^n\|_{G^0} & & 0 \\ & \ddots & \\ 0 & & \|\tilde{y}_N^n\|_{G^0} \end{pmatrix},$$

spectral density f (λ) and spectral density $g(\lambda) = \frac{1}{2\pi |Q(i\lambda)|^2}$ are such, that relation f (λ) /g (λ) H - is admissible [1]).

FOOTNOTE [1]. Comp. with introduced previously condition (3.5), which is satisfied, if relation f (λ) / g (λ) is limited. ENDFOOTNOTE.

Let us establish/install now communication/connection of the spectral measures H (dλ) and H^0 (dλ).

By integrating in parts, it is possible to arrive at the following equality, analogous (4.41):

$$\tilde{y}_k^n(\lambda) = \int_0^T e^{i\lambda t}\left[Q\left(\frac{d}{dt}\right)\theta(t)\right]dt = Q(-i\lambda)\,\tilde{\theta}_k^n(\lambda) + r_k^n(\lambda).$$

where (as was to designation earlier)

$$\tilde{\theta}_k^n(\lambda) = \int_0^T e^{i\lambda t}\theta_k(t)$$

and the "residue/remainder" $r_k^n(\lambda)$ satisfies (under condition (4.37)) the relationship/ratio

$$\|r_k^n\|_G = o\left\{\|\tilde{y}_k^n\|_{G^0}\right\}.$$

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Therefore (under the condition of the weak convergence of sequence M^n $(d\lambda)$

$$\lim_{n\to\infty} \int \frac{1}{|Q(i\lambda)|^2} \frac{|\tilde{y}_k^n(\lambda)|^2}{\|\tilde{y}_k^n\|_{G^0}} G^0(d\lambda) =$$

$$= \lim_{n\to\infty} \int \frac{1}{|Q(i\lambda)|^2} M_{kk}^n(d\lambda) = \int \frac{1}{|Q(i\lambda)|^2} H_{kk}(d\lambda)$$

it is simultaneous

$$\lim_{n\to\infty} \int \frac{1}{|Q(i\lambda)|^2} \frac{|\tilde{y}_k^n(\lambda)|^2}{\|\tilde{y}_k^n\|_{G^0}} G^0(d\lambda) = \lim_{n\to\infty} \frac{\|\tilde{\theta}_k^n\|_{G^0}}{\|\tilde{y}_k^n\|_{G^0}}.$$

It is evident that there is the final and different from 0 limit

$$d_k = \lim_{n\to\infty} \frac{\|\tilde{\theta}_k^n\|_{G^0}}{\|\tilde{y}_k^n\|_{G^0}}, \qquad k = 1, \ldots, N.$$

Furthermore, as can easily be seen,

$$\int \frac{\varphi(\lambda)}{|Q(i\lambda)|^2} H_{kl}(d\lambda) =$$

$$= \lim_{n\to\infty} \int \frac{\varphi(\lambda)}{|Q(i\lambda)|^2} \frac{\tilde{\theta}_k^n(\lambda)\,\overline{\tilde{\theta}_l^n(\lambda)}}{\|\tilde{\theta}_k^n\|_{G^0}\|\tilde{\theta}_l^n\|_{G^0}} G^0(d\lambda) =$$

$$= d_k \left[\lim_{n\to\infty} \int \varphi(\lambda) \frac{\tilde{\theta}_k^n(\lambda)\,\overline{\tilde{\theta}_l^n(\lambda)}}{\|\tilde{\theta}_k^n\|_{G^0}\|\tilde{\theta}_l^n\|_{G^0}} G^0(d\lambda) \right] d_l =$$

$$= d_k \left[\int \varphi(\lambda) H^0(d\lambda) \right] d_l$$

for any limited and continuous function ϕ (λ).

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This indicates that under **the condition for existence** of the spectral measure H (dλ) of functions $y_k(t) = Q\left(\frac{d}{dt}\right)\theta_k(t)$, k = 1, ..., N, there is the spectral measure H° (dλ), also, of original functions θ (t), k = 1, ..., N, whereupon

$$H^0(d\lambda) = d^{-1}\left[\frac{1}{|Q(i\lambda)|^2} H(d\lambda)\right] d^{-1}, \qquad (4.44)$$

where the diagonal matrix/die

$$d = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_N \end{pmatrix}$$

has on diagonal the cell/elements

$$d_k = \lim_{n\to\infty} \frac{\|\theta_k^n\|_{G^0}}{\|\tilde{g}_k^n\|_{G^0}}, \qquad k = 1, \ldots, N.$$

If we now, using relationship/ratio (4.44), in formula (4.42) of H (dλ) pass to H° (dλ), then, after replacing the normalization matrix/die D_n by the matrix/die

$$D_n^0 = \begin{Vmatrix} \|\theta_1^n\|_{G^0} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \|\theta_N^n\|_{G^0} \end{Vmatrix} = dD_n,$$

let us arrive, in particular, at the following result.

Theorem 11. Let spectral density g (λ) = 1/2π $|Q_0(i\lambda)|^2$ be such, that function h (λ) = f (λ) / g (λ) is limited, and spectral density f (λ) is continuous[1]), whereupon there is a nondegenerate spectral measure of functions $\left\{Q\left(\frac{d}{dt}\right)\theta_1(t), \ldots, Q\left(\frac{d}{dt}\right)\theta_N(t)\right\}$, and is also made condition (4.37). Then for a correlation matrix/die $\{\sigma_{kl}^n\}$ best estimators occurs the following asymptotic formula [1]) (comp. (4.25)):

$$\lim_{n\to\infty} D_n^0 \{\sigma_{kl}^n\} D_n^0 = \frac{1}{2\pi}\left[\int \frac{1}{g(\lambda)} H^0(d\lambda)\right]^{-1}\left[\int h(\lambda)\frac{1}{g(\lambda)} H^0(d\lambda)\right]\left[\int \frac{1}{g(\lambda)} H^0(d\lambda)\right]^{-1}. \tag{4.45}$$

where H° (dλ) - the spectral measure of functions $\theta_1(t), \ldots, \theta_n(t)$ u h (λ) = f (λ) / g (λ).

FOOTNOTE [1]. It is necessary to say that formula (4.45) is obtained and common for the positive evenly continuous spectral densities g (λ), for which there are limits $\lim_{\lambda\to\pm\infty} g(\lambda)\lambda^{2m}$. A. S. Kholevo: 1) about the estimations of coefficients regression. Theoretical probability and its applications. XIV No 1 (1969), 78-101. ENDFOOTNOTE.

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Let us note that in the case of a spectral density of the type" f (λ) = 1/2π Q (iλ) this formula gives the following asymptotic expression for a correlation matrix/die $\{s^n_{kl}\}$ the best unbiased estimates:

$$\lim_{n\to\infty} D^0_n \{s^n_{kl}\} D^0_n = 2\pi\left[\int \frac{1}{f(\lambda)} H^0(d\lambda)\right]^{-1} \tag{4.46}$$

(comp. (4.29)).

Let us explain now, when is correctly inequality (4.26), establish/installed earlier in the case of the discrete time t under condition f (λ) $\asymp$ 1. As the analog of this condition can serve following:

$$f(\lambda)\asymp(1+\lambda^2)^{-m}. \tag{4.47}$$

where m is certain positive integer.

Let us turn to the best estimators, constructed according to pseudospectral density $g(\lambda) = 1/2\pi |Q(i\lambda)|^2$, where $Q(z) = (1+lz)^m$, let us assume that there is the spectral measure $H(d\lambda)$ of functions $y_k(t) = Q\left(\frac{d}{dt}\right)\theta_k(t)$, k = 1, ..., N.

FOOTNOTE [2]. On the asymptotic efficiency of best estimators. theoretical probability and its applications. (in press). ENDFOOTNOTE.

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Then in perfect analogy with that, as this was done equal at the conclusion of inequality (4.26), we will obtain that for correlation matrix/die $\{\sigma^n_{kl}\}$ any linear unbiased estimates $\tilde{\alpha}^n_1, \ldots, \tilde{\alpha}^n_N$ occurs the inequality

$$\{\sigma^n_{kl}\} \geqslant \left\{ \int \frac{g(\lambda)}{f(\lambda)} \psi^n_k(\lambda)\, \overline{\psi^n_k(\lambda)}\, G(d\lambda) \right\}^{-1}$$

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(let us note that with condition (4.47) function g (λ) / f (λ) is limited, more precise, g (λ) / f (λ) $\asymp$ 1). Using common/general/total relationship/ratio (4.42), is hence concluded, that in the case of the continuous spectral density f (λ)

$$\lim_{n\to\infty} D_n \{\sigma_{kl}^n\} D_n \geqslant$$
$$\geqslant \lim_{n\to\infty} \left\{ \int \frac{g(\lambda)}{f(\lambda)} \frac{\psi_k^n(\lambda)\overline{\psi_l^n(\lambda)}}{\|\tilde{\psi}_k^n\|_{G^0}\|\tilde{\psi}_l^n\|_{G^0}} G(d\lambda) \right\}^{-1} = \left[\int \frac{g(\lambda)}{f(\lambda)} H(d\lambda) \right]^{-1}.$$

Passing here of H (dλ) to H^0 (dλ) (see (4.44) and taking into account that g (λ) = 1/2π Q (iλ) $|^2$, we will obtain an inequality of type (4.26), namely:

$$\lim_{n\to\infty} D_n^0 \{\sigma_{kl}^n\} D_n^0 \geqslant 2\pi \left[\int \frac{1}{f(\lambda)} H^0(d\lambda) \right]^{-1}.$$

Let us assume now that spectral density f (λ) instead of (4.47) satisfies only the condition

$$f(\lambda) \geqslant k(1+\lambda^2)^{-n} \qquad (4.48)$$

(where k is certain positive constant). Let us introduce auxiliary spectral density

$$f_r(\lambda) = \min\{r(1+\lambda^2)^{-m},\ f(\lambda)\}.$$

It is obvious, $f_r(\lambda)$ satisfies condition (4.47), and if $\{\sigma_{kl}^n\}_r$ is a correlation matrix/die of the same linear unbiased estimates $\tilde{\alpha}_1^n, \ldots, \tilde{\alpha}_N^n$ (but calculated relative to spectral density $f_r(\lambda)$), that according to demonstrated recently the inequality

$$\lim_{n\to\infty} D_n^0 \{\sigma_{kl}^n\}_r D_n^0 \geqslant 2\pi \left[\int \frac{1}{f_r(\lambda)} H^0(d\lambda) \right]^{-1}.$$

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Since $f(\lambda) \geqslant f_r(\lambda)$, that, obviously,

$$\{\sigma_{kl}^n\} \geqslant \{\sigma_{kl}^n\}_r$$

and

$$\lim_{n\to\infty} D_n^0 \{\sigma_{kl}^n\} D_n^0 \geqslant 2\pi \left[\int \frac{1}{f_r(\lambda)} H^0(d\lambda) \right]^{-1}$$

with any r > 0. But with r -> ∞ the monotonically descending sequence of functions $1/f_r(\lambda)$ has as its limit 1/f (λ), so that

$$\lim_{r\to\infty} \int \frac{1}{f_r(\lambda)} H^0(d\lambda) = \int \frac{1}{f(\lambda)} H^0(d\lambda)$$

and

$$\lim_{n\to\infty} D_n^0 \{\sigma_{kl}^n\} D_n^0 \geqslant 2\pi \left[\int \frac{1}{f(\lambda)} H^0(d\lambda) \right]^{-1} \qquad (4.49)$$

(where for the limited spectral density f (λ) under the condition of the nondegeneracy of matrix/die H° = ∫ H° (dλ) will be nondegenerate matrix/die $\int \frac{1}{f(\lambda)} H^0(d\lambda)$).

Thus, we arrived at the following result.

Theorem 12. Let spectral density f (λ) satisfy condition (4.48) and is continuous, there is a nondegenerate spectral measure of functions $\left\{Q\left(\frac{d}{dt}\right)\theta_1(t), \ldots, Q\left(\frac{d}{dt}\right)\theta_N(t)\right\}$, where $Q\left(\frac{d}{dt}\right) = \left(1+\frac{d}{dt}\right)^m$, and is also made condition (4.37). Then for a correlation matrix/die $\{\sigma_{kl}^n\}$ any linear unbiased estimates occurs asymptotic inequality (4.49), in which H°

$\{\theta_1(t), \ldots, \theta_N(t)\}.$

(dλ) - also the nondegenerate spectral measure of original functions

Of course, just as in the case of discrete time (see theorem 8), under the condition of validity of relationship/ratios (4.45) and (4.49)

for the asymptotic effectiveness of the best estimators, constructed according to spectral density g (λ), it is sufficient in order that matrix function f (λ) / g (λ) E would be constant almost everywhere relative to the spectral matrix measure H^0 (dλ); under further condition (4.46) this also is necessary for a efficiency.

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Example. Let N = 2 and

$$\theta_1(t) = P_1(t)\int e^{i\lambda t} m_1(d\lambda), \quad \theta_2(t) = P_2(t)\int e^{i\lambda t} m_2(d\lambda),$$

where P_1 (t) and P_2 (t) - polynomials of t of degree n_1 and n_2 respectively, of which the coefficients with leading terms are equal to 1, m_i (dλ) - the final measures.

Let us designate Λ_1 and Λ_2 the set of the points λ, for which $m_1(\lambda) \neq 0$ and m_2 (λ) $\neq$ 0 respectively. It is easy to count, that

the spectral measure H^0 $(d\lambda)$ = $\{H_{kl}^0(d\lambda)\}$ functions $\{\theta_1(t),\ \theta_2(t)\}$ is purely discrete and concentrating isolated points $\lambda \in \Lambda_1 \cup \Lambda_2$, more precise[1]),

$$H^0(\lambda) = d^{-1}\begin{pmatrix} \frac{1}{2n_1+1}|m_1(\lambda)|^2 & \frac{1}{n_1+n_2+1} m_1(\lambda)\, m_2(\lambda) \\ \frac{1}{n_1+n_2+1}\overline{m_1(\lambda)}\, m_2(\lambda) & \frac{1}{2n_2+1}|m_2(\lambda)|^2 \end{pmatrix} d^{-1},$$

where d is a diagonal matrix/die of the form

$$d = \begin{vmatrix} \sqrt{\frac{1}{2n_1+1}\sum |m_1(\lambda)|^2} & 0 \\ 0 & \sqrt{\frac{1}{2n_2+1}\sum |m_2(\lambda)|^2} \end{vmatrix}.$$

FOOTNOTE 1. See, for example, [28]. ENDFOOTNOTE.

When sets Λ_1 and Λ_2 do not intersect, the condition: f (λ) / g (λ) E is constant almost everywhere relative to H^0 $(d\lambda)$ - equivalently to the fact that ratio f (λ) / g (λ) is constant for each of the "cell/elements of spectrum" Λ_1 and Λ_2. This condition, obviously, is always satisfied in the case, when there are only unique points $\lambda_1 \in \Lambda_1$ и $\lambda_2 \in \Lambda_2$ and In the case, when the spectrum $\Lambda_1 \cup \Lambda_2$ consists of a finite number of points $\lambda_1, \ldots, \lambda_m$, the indicated condition of asymptotic efficiency is made, if we take pseudospectral density g (λ) of type (4.35) such, that g (λ) = f (λ) with $\lambda = \lambda_1, \ldots, \lambda_m$.

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REFERENCES

1. Ахиезер Н. И., Лекции по теории аппроксимации, М., 1965.
2. Ахиезер Н. И., Глазман И. М., Теория линейных операторов в гильбертовом пространстве, М., 1966.
3. Бари Н. К., Тригонометрические ряды, М., 1961.
4. Бари Н. К., Биортогональные системы и базисы в гильбертовом пространстве, Уч. зап. МГУ, сер. матем., 4 (1961), 69–107.
5. Гельфанд И. М., Виленкин Н. Я., Обобщенные функции, IV. Некоторые применения гармонического анализа. Оснащенные гильбертовы пространства, М., 1961.
6. Гельфанд И. М., Шилов Г. Е., Обобщенные функции и действия над ними, М., 1958.
7. Гельфонд А. О., Исчисление конечных разностей, М.–Л., 1952.
8. Гихман И. И., Скороход А. В., Введение в теорию случайных процессов М., 1965.
9. Голузин Г. М., Геометрическая теория функций комплексного переменного, М., 1957.
10. Гофман К., Банаховы пространства аналитических функций (перев. с англ.), М., 1963.
11. Гренадер У., Сеге Г., Теплицевы формулы и их приложения (перев. с англ.), М., 1961.
11. Дуб Дж. Л., Вероятностные процессы (перев. с англ.), М., 1956.
13. *Зигмунд А., Тригонометрические ряды (перев. с англ.)*, тт. I, II, М., 1965.
14. Ибрагимов И. А., Линник Ю. В., Независимые и стационарно связанные величины, М., 1965.
15. Крылов В. И., О функциях, регулярных в полуплоскости, Матем. сб. 4, № 46 (1938), 9–30.
16. Левин Б. Я., Распределение корней целых функций, М., 1956.
17. Леман Э., Проверка статистических гипотез (перев. с англ.), М., 1967.
18. Маркушевич А. И., Теория аналитических функций, М., 1950.
19. Пинскер М. С., Информация и информационная устойчивость случайных величин и процессов, М., 1960.
20. Привалов И. И., Граничные свойства аналитических функций, М.–Л., 1950.
21. Понтрягин Л. С., Обыкновенные дифференциальные уравнения, М., 1961.
22. Розанов Ю. А., Стационарные случайные процессы, М., 1963.
23. Розанов Ю. А., Гауссовские бесконечномерные распределения, М., 1968.
24. Сеге Г., Ортогональные многочлены (перев. с англ.), М., 1962.
25. Тиман А. Ф., Теория приближения функций действительного переменного, М., 1960.
26. Хилл Э., Функциональный анализ и полугруппы (перев. с англ.), М., 1951.
27. Cramer H., Leadbetter M., Stationary and related stochastic processes, N. Y.—London, 1967.
28. Grenander V., Rosenblatt M., Statistical analyses of stationary time series, N. Y., 1956.

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